Quantum field theory I	**Exercise sheet 3**

Canonical quantization of the harmonic oscillator

Exercise 1: We consider the harmonic oscillator (in units where $\hbar = 1$), with

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{m\omega^2\hat{x}^2}{2} \;, \quad \hat{a} = \sqrt{\frac{m\omega}{2}}\hat{x} + \frac{i}{\sqrt{2m\omega}}\,\hat{p}\;, \quad \hat{a}^\dagger = \sqrt{\frac{m\omega}{2}}\hat{x} - \frac{i}{\sqrt{2m\omega}}\,\hat{p}\;.$$

Assuming $[\hat{x},\hat{p}] = i$ and $[\hat{x},\hat{x}] = [\hat{p},\hat{p}] = 0$, show that

(a) $[\hat{a},\hat{a}] = [\hat{a}^\dagger,\hat{a}^\dagger] = 0\;, \quad [\hat{a},\hat{a}^\dagger] = 1\;.$

(b) $\hat{H} = \omega\Big(\hat{a}^\dagger\hat{a} + \frac{1}{2}\Big)\;.$

(c) $\hat{N} \equiv \hat{a}^\dagger\hat{a}$ is Hermitean, and has real eigenvalues.

(d) $\hat{N}|n\rangle \equiv n|n\rangle$ and $\langle n|n\rangle \equiv 1 \Rightarrow \hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle\;, \quad \hat{a}|n\rangle = \sqrt{n}|n-1\rangle\;.$

Exercise 2: In the script we have determined $\hat{x}_H(t)$ for the harmonic oscillator, starting from the equations of motion. Now we do the same more directly.

(a) Verify that

$$e^{\hat{A}}\hat{B}e^{-\hat{A}} = \hat{B} + [\hat{A},\hat{B}] + \frac{1}{2!}[\hat{A},[\hat{A},\hat{B}]] + \frac{1}{3!}[\hat{A},[\hat{A},[\hat{A},\hat{B}]]] + \ldots\;.$$

(b) With the help of this expansion, determine the operator

$$\hat{x}_H(t) \;\equiv\; e^{i\hat{H}t}\,\hat{x}\,e^{-i\hat{H}t}\;.$$