Beispiele: (i)

$$f(x) = A\, e^{-\frac{(x-x_0)^2}{\Delta^2}} \quad ; \quad \Delta > 0 \qquad \text{„Gaußsches Wellenpaket"}$$

$$\tilde{f}(k) = \int_{-\infty}^{\infty} dx\, f(x)\, e^{-ikx} = A \int_{-\infty}^{\infty} dx\, e^{-\left[\frac{(x-x_0)^2}{\Delta^2} + ikx\right]}$$

$$= A \int_{-\infty}^{\infty} dx\, \exp\left[-\frac{1}{\Delta^2}\left(x^2 - 2x_0 x + x_0^2 + ik\Delta^2 x\right)\right]$$

$$= A \cdot \int_{-\infty}^{\infty} dx\, \exp\left[-\frac{1}{\Delta^2}\left(x^2 - 2x\left[x_0 - \frac{ik\Delta^2}{2}\right] + x_0^2\right)\right]$$

$$= A \cdot \int_{-\infty}^{\infty} dx\, \exp\left[-\frac{1}{\Delta^2}\left(\left(x - x_0 + \frac{ik\Delta^2}{2}\right)^2 + \underbrace{x_0^2 - \left(x_0 - \frac{ik\Delta^2}{2}\right)^2}_{\cancel{x_0^2} - \cancel{x_0^2} + x_0 ik\Delta^2 + \frac{k^2\Delta^4}{4}}\right)\right]$$

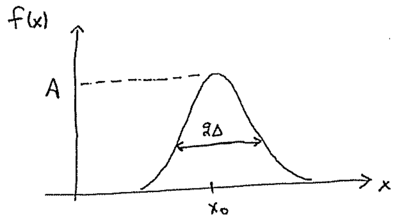
$$= A \cdot e^{-\frac{k^2\Delta^2}{4} - ikx_0} \cdot \underbrace{\int_{-\infty}^{\infty} dx\, \exp\left[-\frac{1}{\Delta^2}\left(x - x_0 + \frac{ik\Delta^2}{2}\right)^2\right]}$$

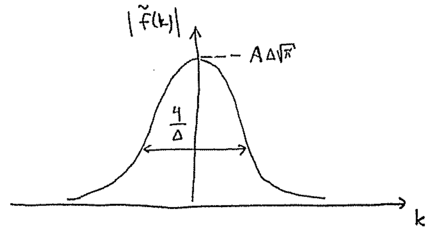
Darf man hier $x = x' + x_0 - \frac{ik\Delta^2}{2}$ substituieren, trotz „i"? Vorlesung MMP III → ja!

$$= A \cdot e^{-\frac{k^2\Delta^2}{4} - ikx_0} \underbrace{\int_{-\infty}^{\infty} dx'\, e^{-\left(\frac{x'}{\Delta}\right)^2}}$$

$y' = \frac{x'}{\Delta}$; $dx' = \Delta dy'$; Seite 25 ⇒ $\Delta \cdot \sqrt{\pi}$

$$= \underline{A\Delta\sqrt{\pi}\, e^{-\frac{k^2\Delta^2}{4} - ikx_0}} \quad ; \quad |\tilde{f}(k)| = A\Delta\sqrt{\pi}\, e^{-\frac{k^2\Delta^2}{4}}$$





Physikalisch: „<u>Unschärferelation</u>":

Wellenpaket { schmal im x-Raum ⇒ { breit im k-Raum !
{ breit —〃— ⇒ { schmal —〃—

(Bemerkungen: * $A \to \frac{1}{\Delta\sqrt{\pi}}$, $x_0 \to 0$ ⇒ $f(x) = \frac{1}{\Delta\sqrt{\pi}} e^{-\frac{x^2}{\Delta^2}}$, $\tilde{f}(k) = e^{-\frac{k^2\Delta^2}{4}}$

$$\Rightarrow \int_{-\infty}^{\infty} dx\, f(x) = \tilde{f}(0) = 1 \qquad \Rightarrow \text{Kap. 3.4}$$

* $$\int_{-\infty}^{\infty} dx\, [f(x)]^2 = A^2 \int_{-\infty}^{\infty} dx\, e^{-\frac{2(x-x_0)^2}{\Delta^2}} = A^2 \cdot \Delta\sqrt{\frac{\pi}{2}}$$

$$\int_{-\infty}^{\infty} \frac{dk}{2\pi} |\tilde{f}(k)|^2 = \frac{A^2\Delta^2}{2} \int_{-\infty}^{\infty} dk\, e^{-\frac{k^2\Delta^2}{2}} = \frac{A^2\Delta^2}{2} \cdot \frac{\sqrt{2\pi}}{\Delta} = A^2\Delta\sqrt{\frac{\pi}{2}} \quad !$$

⇒ Kap. 3.4)

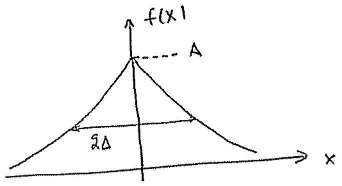
(ii) $f(x) = A\, e^{-\frac{|x|}{\Delta}}$; $\Delta > 0$

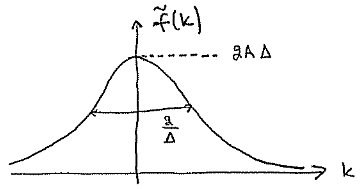
$$\tilde{f}(k) = \int_{-\infty}^{\infty} dx\, f(x)\, e^{-ikx} = A\left\{ \int_0^{\infty} dx\, e^{-\frac{x}{\Delta} - ikx} + \underbrace{\int_{-\infty}^{0} dx\, e^{\frac{x}{\Delta} - ikx}}_{\text{substituiere } x=-x' \quad dx = -dx' \;;\; \text{nachher } x' \to x} \right\}$$

$$= A \int_0^{\infty} dx \left\{ \exp\left[-x\left(\tfrac{1}{\Delta} + ik\right)\right] + \exp\left[-x\left(\tfrac{1}{\Delta} - ik\right)\right] \right\}$$

$$= A \cdot \left\{ \frac{-1}{\frac{1}{\Delta} + ik} \left[e^{-x\left(\frac{1}{\Delta}+ik\right)}\right]_0^{\infty} + \frac{-1}{\frac{1}{\Delta} - ik} \left[e^{-x\left(\frac{1}{\Delta}-ik\right)}\right]_0^{\infty} \right\}$$

$$= A \left\{ \frac{1}{\frac{1}{\Delta} + ik} + \frac{1}{\frac{1}{\Delta} - ik} \right\} = \frac{2A\,\Delta^{-1}}{k^2 + (\Delta^{-1})^2}$$





(iii) In drei Dimensionen:

* $\tilde{f}(\vec{k}) = \frac{1}{k^2}$, $k := |\vec{k}|$

$$\Longrightarrow \quad f(\vec{r}) = \int \frac{d^3\vec{k}}{(2\pi)^3} \frac{e^{i\vec{k}\cdot\vec{r}}}{k^2}$$

Benutze Kugelkoordinaten: $d^3\vec{k} \to dk\, k^2\, d\theta \sin\theta\, d\varphi$

Sei $\theta = \sphericalangle(\vec{k}, \vec{r})$. Seite 32 $\Rightarrow$ $\int_0^{\pi} d\theta \sin\theta\, g(\cos\theta) = \int_{-1}^{+1} dz\, g(z)$

$$= \frac{2\pi}{(2\pi)^3} \int_0^{\infty} dk \int_{-1}^{+1} dz\, e^{ikrz} \qquad \left(2\pi = \int_0^{2\pi} d\varphi\right)$$

$$= \frac{1}{4\pi^2} \int_0^{\infty} \frac{dk}{ikr} \left(e^{ikr} - e^{-ikr}\right) = \frac{1}{2\pi^2 r} \underbrace{\int_0^{\infty} \frac{dk}{k} \sin(kr)}_{\int_0^{\infty} \frac{dt}{t} \sin(t) = \frac{\pi}{2}}$$

(Aufgabe 12.3)

$$= \frac{1}{4\pi r}$$

* $f(\vec{r}) = \frac{1}{4\pi r}$, $r := |\vec{r}|$

$$\Longrightarrow \quad \tilde{f}(\vec{k}) = \int d^3\vec{r}\, \frac{e^{-i\vec{k}\cdot\vec{r}}}{4\pi r}$$

Kugelkoordinaten: $d^3\vec{r} \to dr\, r^2\, d\theta \sin\theta\, d\varphi$

$$= \frac{2\pi}{4\pi} \int_0^{\infty} dr\, r \int_{-1}^{+1} dz\, e^{-ikrz}$$

$$= \frac{1}{2} \int_0^{\infty} \frac{dr}{-ik} \left(e^{-ikr} - e^{ikr}\right) \overset{r = \frac{x}{k}}{=} \frac{1}{2k^2} \int_0^{\infty} dx\, \frac{e^{ix} - e^{-ix}}{i}$$

Das Integral ist nicht wirklich wohldefiniert, aber mittels eines angemessenen Limes geht es schon:

$$\lim_{\varepsilon \to 0^+} \int_0^{\infty} \frac{dx}{i} \left[e^{ix - \varepsilon x} - e^{-ix - \varepsilon x}\right] = \lim_{\varepsilon \to 0^+} \frac{1}{i} \left\{ \left[\frac{e^{x(i-\varepsilon)}}{i-\varepsilon}\right]_0^{\infty} + \left[\frac{e^{-x(i+\varepsilon)}}{i+\varepsilon}\right]_0^{\infty} \right\}$$

$$= \lim_{\varepsilon \to 0^+} \frac{1}{i} \left\{ -\frac{1}{i-\varepsilon} - \frac{1}{i+\varepsilon} \right\} = \frac{-2}{i^2} = +2$$

$$\Rightarrow \quad \tilde{f}(\vec{k}) = \frac{1}{k^2} \qquad \text{ok!}$$