Divergenz

$$\vec{E} = E_u \vec{e}_u + E_v \vec{e}_v + E_w \vec{e}_w$$

Die Schwierigkeit: $\nabla \cdot \vec{E} = \vec{e}_u \cdot \frac{1}{h_u} \frac{\partial}{\partial u} \left(E_u \vec{e}_u + E_v \vec{e}_v + E_w \vec{e}_w \right) + \ldots$

! !

wegen $\vec{e}_u \cdot \vec{e}_v = 0$ oder $\vec{e}_u \cdot \partial_u \vec{e}_u = \frac{1}{2} \partial_u (\vec{e}_u \cdot \vec{e}_u) = 0$

Trick: drücke $\nabla \cdot \vec{E}$ mittels bekannter geometrischer Größen aus!

Seite 12 $\rightsquigarrow$

$$\nabla \cdot \vec{E} = \lim_{dx \to 0} \frac{1}{dx} \left\{ E_x \left(\vec{r} + \tfrac{1}{2} dx \vec{e}_x \right) - E_x \left(\vec{r} - \tfrac{1}{2} dx \vec{e}_x \right) \right\} + \ldots$$

$$= \lim_{dx \to 0} \frac{1}{dx\,dy\,dz} \left\{ dy\,dz\, E_x \left(\vec{r} + \frac{dx \vec{e}_x}{2} \right) - dy\,dz\, E_x \left(\vec{r} - \frac{dx \vec{e}_x}{2} \right) \right\} + \ldots$$

$$= \lim_{d\vec{r} \to 0} \frac{1}{dV} \left\{ d\vec{A} \cdot \vec{E} \left(\vec{r} + \frac{d\vec{r}}{2} \right) - d\vec{A} \cdot \vec{E} \left(\vec{r} - \frac{d\vec{r}}{2} \right) \right\} + \ldots$$

$d\vec{r} \to du\, \partial_u \vec{r}$
$d\vec{A} \to dv\,dw\, \partial_v \vec{r} \times \partial_w \vec{r} = dv\,dw\, h_v h_w\, \vec{e}_v \times \vec{e}_w = dv\,dw\, h_v h_w\, \vec{e}_u$
$dV \to du\,dv\,dw \left| \partial_u \vec{r} \cdot (\partial_v \vec{r} \times \partial_w \vec{r}) \right| = du\,dv\,dw\, h_u h_v h_w \left| \vec{e}_u \cdot (\vec{e}_v \times \vec{e}_w) \right|$

$$\stackrel{!}{=} \lim_{du \to 0} \frac{1}{du\, h_u h_v h_w} \left\{ h_v h_w E_u \left(\vec{r}\left(u + \tfrac{du}{2}\right) \right) - h_v h_w E_u \left(\vec{r}\left(u - \tfrac{du}{2}\right) \right) \right\} + \ldots$$

D.h.: $$\boxed{\nabla \cdot \vec{E} = \frac{1}{h_u h_v h_w} \left\{ \partial_u (h_v h_w E_u) + \partial_v (h_w h_u E_v) + \partial_w (h_u h_v E_w) \right\}}\,.$$

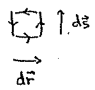
Beispiel: Kugelkoordinaten: (Seite 28 mit $u \to r,\ v \to \theta,\ w \to \varphi$)

$$\nabla \cdot \vec{E} = \frac{1}{r^2} \partial_r (r^2 E_r) + \frac{1}{r \sin\theta} \partial_\theta (\sin\theta\, E_\theta) + \frac{1}{r \sin\theta} \partial_\varphi E_\varphi$$

Für $\vec{E} \to \vec{r} = r \vec{e}_r \Rightarrow \nabla \cdot \vec{E} = \frac{1}{r^2} \partial_r (r^3) = 3$ (vgl. Seite 12).

Rotation

Seite 13 $\rightsquigarrow$



$$(\nabla \times \vec{E})_z = \lim_{dx \to 0} \frac{1}{dx} \left\{ E_y \left(\vec{r} + \tfrac{1}{2} dx \vec{e}_x \right) - E_y \left(\vec{r} - \tfrac{1}{2} dx \vec{e}_x \right) \right\} + \ldots$$

$$= \lim_{dx \to 0} \frac{1}{dx\,dy} \left\{ dy\, E_y \left(\vec{r} + \frac{dx \vec{e}_x}{2} \right) - dy\, E_y \left(\vec{r} - \frac{dx \vec{e}_x}{2} \right) \right\} + \ldots$$

$$= \lim_{d\vec{r} \to 0} \frac{1}{|d\vec{A}|} \left\{ d\vec{s} \cdot \vec{E} \left(\vec{r} + \frac{d\vec{r}}{2} \right) - d\vec{s} \cdot \vec{E} \left(\vec{r} - \frac{d\vec{r}}{2} \right) \right\} + \ldots$$

$d\vec{r} \to du\, \partial_u \vec{r}$, $d\vec{s} \to dv\, h_v \vec{e}_v$
$d\vec{A} \to du\,dv\, h_u h_v\, \vec{e}_u \times \vec{e}_v$

$$\stackrel{!}{=} \lim_{du \to 0} \frac{1}{du\, h_u h_v} \left\{ h_v E_v \left(\vec{r}\left(u + \tfrac{du}{2}\right) \right) - h_v E_v \left(\vec{r}\left(u - \tfrac{du}{2}\right) \right) \right\} + \ldots$$

D.h.: $$\boxed{\nabla \times \vec{E}} = \frac{\vec{e}_w}{h_u h_v} \left\{ \partial_u (h_v E_v) - \partial_v (h_u E_u) \right\} + \ldots = \boxed{\frac{1}{h_u h_v h_w} \begin{vmatrix} h_u \vec{e}_u & h_v \vec{e}_v & h_w \vec{e}_w \\ \partial_u & \partial_v & \partial_w \\ h_u E_u & h_v E_v & h_w E_w \end{vmatrix}}$$

Beispiel: Zylinderkoordinaten: $u \to \rho,\ v \to \varphi,\ w \to z$

$h_\rho = 1,\quad h_\varphi = \rho,\quad h_z = 1$

$$\nabla \times \vec{E} = \frac{1}{\rho} \begin{vmatrix} \vec{e}_\rho & \rho \vec{e}_\varphi & \vec{e}_z \\ \partial_\rho & \partial_\varphi & \partial_z \\ E_\rho & \rho E_\varphi & E_z \end{vmatrix}$$

Für $\vec{E} = \vec{\omega} \times \vec{r}$; $\vec{\omega} := \omega \vec{e}_z$; $\vec{r} = \rho \vec{e}_\rho + z \vec{e}_z$

$\vec{\omega} \times \vec{r} = \omega \rho\, \vec{e}_z \times \vec{e}_\rho = \omega \rho\, \vec{e}_\varphi$

$$\Rightarrow \nabla \times \vec{E} = \frac{1}{\rho} \begin{vmatrix} \vec{e}_\rho & \rho \vec{e}_\varphi & \vec{e}_z \\ \partial_\rho & \partial_\varphi & \partial_z \\ 0 & \omega \rho^2 & 0 \end{vmatrix}$$

$$= \frac{\vec{e}_z}{\rho} \partial_\rho (\omega \rho^2) = 2 \omega \vec{e}_z = 2 \vec{\omega}$$

(vgl. Seite 13).

Laplace-Operator in Kugelkoordinaten

(i) Aus den Formeln für Nabla (Seite 28) und Divergenz (Seite 29):

$$\nabla^2 = \nabla \cdot \nabla = \frac{1}{h_u h_v h_w} \left\{ \partial_u \left[\frac{h_v h_w}{h_u} \partial_u \right] + \partial_v \left[\frac{h_w h_u}{h_v} \partial_v \right] + \partial_w \left[\frac{h_u h_v}{h_w} \partial_w \right] \right\}$$

Seite 28:
$u \to r$; $h_u \to h_r = 1$
$v \to \theta$; $h_v \to h_\theta = r$
$w \to \varphi$; $h_w \to h_\varphi = r \sin\theta$

$$= \frac{1}{r^2 \sin\theta} \left\{ \partial_r \left[r^2 \sin\theta \, \partial_r \right] + \partial_\theta \left[\sin\theta \, \partial_\theta \right] + \partial_\varphi \left[\frac{1}{\sin\theta} \partial_\varphi \right] \right\}$$

$$= \frac{1}{r^2} \partial_r \left[r^2 \partial_r \right] + \frac{1}{r^2 \sin\theta} \partial_\theta \left[\sin\theta \, \partial_\theta \right] + \frac{1}{r^2 \sin^2\theta} \partial_\varphi^2$$

$$= \frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \left[\frac{\partial^2}{\partial \theta^2} + \frac{1}{\tan\theta} \frac{\partial}{\partial \theta} + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial \varphi^2} \right].$$

(ii) Explizit: Seite 28. $\Rightarrow$

$$\vec{e}_r = \begin{pmatrix} \sin\theta \cos\varphi \\ \sin\theta \sin\varphi \\ \cos\theta \end{pmatrix}, \quad \vec{e}_\theta = \begin{pmatrix} \cos\theta \cos\varphi \\ \cos\theta \sin\varphi \\ -\sin\theta \end{pmatrix}, \quad \vec{e}_\varphi = \begin{pmatrix} -\sin\varphi \\ \cos\varphi \\ 0 \end{pmatrix}$$

$$\nabla = \vec{e}_r \frac{\partial}{\partial r} + \vec{e}_\theta \frac{1}{r} \frac{\partial}{\partial \theta} + \vec{e}_\varphi \frac{1}{r \sin\theta} \frac{\partial}{\partial \varphi}$$

Es folgt:

$$\partial_r \vec{e}_r = \partial_r \vec{e}_\theta = \partial_r \vec{e}_\varphi = 0$$

$$\partial_\theta \vec{e}_r = \vec{e}_\theta \; ; \quad \partial_\theta \vec{e}_\theta = -\vec{e}_r \; ; \quad \partial_\theta \vec{e}_\varphi = 0 .$$

$$\partial_\varphi \vec{e}_r = \sin\theta \, \vec{e}_\varphi \; ; \quad \partial_\varphi \vec{e}_\theta = \cos\theta \, \vec{e}_\varphi \; ;$$

$$\partial_\varphi \vec{e}_\varphi = -\sin\theta \, \vec{e}_r - \cos\theta \, \vec{e}_\theta$$

$$\Rightarrow \quad \nabla \cdot \nabla = \vec{e}_r \cdot \frac{\partial}{\partial r} \left(\vec{e}_r \frac{\partial}{\partial r} + \vec{e}_\theta \frac{1}{r} \frac{\partial}{\partial \theta} + \vec{e}_\varphi \frac{1}{r \sin\theta} \frac{\partial}{\partial \varphi} \right)$$

$$+ \vec{e}_\theta \cdot \frac{1}{r} \frac{\partial}{\partial \theta} \left(\quad \text{—}\|\text{—} \quad \right)$$

$$+ \vec{e}_\varphi \cdot \frac{1}{r \sin\theta} \frac{\partial}{\partial \varphi} \left(\quad \text{—}\|\text{—} \quad \right)$$

$$= \frac{\partial^2}{\partial r^2}$$

$$+ \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2}$$

$$+ \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\cos\theta}{\sin\theta} \frac{\partial}{\partial \theta} + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2}{\partial \varphi^2}$$

$$= \frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \left[\frac{\partial^2}{\partial \theta^2} + \frac{1}{\tan\theta} \frac{\partial}{\partial \theta} + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial \varphi^2} \right] .$$