exercises for mechanics I — sheet 8

[tutorials 25.4. and 29.4., hand-in 6.5., solutions please in English]

Exercise 1 (one-dimensional crystal): The masses $m_n = m$, $n = 0, \pm 1, \pm 2, \ldots$ can move along the $\tilde{x}$-axis. Harmonic springs (spring constant k) between neighbouring masses establish the rest positions $\tilde{x}_n^0 = an$; the length a is called the lattice constant of this model. The deviations from the rest positions are denoted by $x_n(t) = \tilde{x}_n(t) - \tilde{x}_n^0$.

(a) Write down the equations of motion for this linear chain. (2 points) [Answer: $m\ddot{x}_n = k(x_{n+1} - 2x_n + x_{n-1})$.]

(b) Show that the equations of motion can be solved with the ansatz $x_n(t) = Q_q(t)e^{iqna}$. (2 points)

(c) Sketch the eigenfrequencies $\omega(q)$ of the normal modes $Q_q(t)$ as a function of q. (2 points) [This dependence is known as a dispersion relation.]

Exercise 2: The deviation of a string from its rest position, denoted by $u(t,x)$ with $0 \le x \le L$, fulfills the wave equation and the boundary conditions

$$(\partial_t^2 - c^2\partial_x^2)u(t,x) = 0 \,, \quad u(t,0) = u(t,L) = 0 \,.$$

Determine the solution $u(t,x)$ for the initial conditions $u(0,x) = A\sin(2\pi x/L)$, $\dot{u}(0,x) = 0$, by representing the x-dependence as a Fourier series. (6 points)

Exercise 3: (2 points each)

(a) Show that the wave equation $(\partial_t^2 - c^2\partial_x^2)u(t,x) = 0$ can be solved with the ansatz $u(t,x) = f(x - ct) + g(x + ct)$, where f and g are arbitrary functions.

(b) We assume the initial conditions $u(0,x) = \alpha(x)$, $\dot{u}(0,x) = \beta(x)$. How must f and g be chosen in order to satisfy these?

(c) Present a solution to Exercise 2 through the determination of the functions f and g.

Exercise 4: The hydrodynamic equations for a one-dimensional flow with mass density $\rho(t,x)$, pressure $p = p(\rho)$ and flow velocity $v_x(t,x)$ read

$$\begin{aligned} \partial_t\rho + \partial_x(\rho v_x) &= 0 \,, \\ \rho\,(\partial_t + v_x\partial_x)\,v_x &= -\partial_x p \,. \end{aligned}$$

We consider a linear perturbation around rest ($\rho = \rho_0, v_x = 0$), i.e. $\rho(t,x) = \rho_0 + \hat{\rho}(t,x)$, $v_x(t,x) = \hat{v}(t,x)$, where $\hat{\rho}$ and $\hat{v}$ are assumed small. Show that to first order in small quantities, the density perturbation satisfies the equation

$$(\partial_t^2 - c_s^2\partial_x^2)\hat{\rho} = 0 \,.$$

(6 points) [Here $c_s^2 \equiv p'(\rho_0)$ is the speed of sound squared.]