9.3 Thermally modified phase space integrals

In order to proceed with eq. (224), we define

$$I_{\rm B(F)}(k) \equiv \int \frac{{\rm d}^3\vec{p}_1}{(2\pi)^3 2p_1} \int \frac{{\rm d}^3\vec{p}_2}{(2\pi)^3 2p_2} (2\pi)^4 \delta^{(4)}(\mathcal{K} - \mathcal{P}_1 - \mathcal{P}_2)\, \mathcal{N}_{\rm B(F)}(p_1, p_2) \;, \tag{225}$$

$$\mathcal{N}_{\rm B}(p_1,p_2) \equiv 1 + n_{\rm B}(p_1) + n_{\rm B}(p_2) \;, \quad \mathcal{N}_{\rm F}(p_1,p_2) \;\equiv\; 1 - n_{\rm F}(p_1) - n_{\rm F}(p_2) \;. \tag{226}$$

We can then carry out the integral over $\vec{p}_2$:

$$\begin{aligned} I_{\rm B(F)}(k) &= \int \frac{{\rm d}^3\vec{p}_1}{(2\pi)^3 2p_1} \int \frac{{\rm d}^3\vec{p}_2}{(2\pi)^3 2p_2} (2\pi)^3 \delta^{(3)}(\vec{k} - \vec{p}_1 - \vec{p}_2)\, (2\pi)\delta(E_k - p_1 - p_2) \mathcal{N}_{\rm B(F)}(p_1,p_2) \\ &= \frac{1}{(4\pi)^2} \int \frac{{\rm d}^3\vec{p}_1}{p_1 |\vec{k} - \vec{p}_1|}\, \delta\Big(E_k - p_1 - |\vec{k} - \vec{p}_1|\Big) \mathcal{N}_{\rm B(F)}(p_1, E_k - p_1) \\ &= \frac{1}{8\pi} \int_0^\infty \frac{{\rm d}p_1\, p_1}{|\vec{k} - \vec{p}_1|} \int_{-1}^{+1} {\rm d}z\, \delta\Big(E_k - p_1 - \sqrt{k^2 + p_1^2 - 2kp_1 z}\Big) \mathcal{N}_{\rm B(F)}(p_1, E_k - p_1) \;, \end{aligned} \tag{227}$$

where spherical coordinates were introduced in the last step and $z = \cos\theta$. This can be interpreted as describing how QED particles of energies p_1 and $E_k - p_1$ "inverse decay" (i.e. pair produce) into a scalar of energy E_k.

Exercise 9.9. The integrations in eq. (227) can be performed in closed form. Show that

$$I_{\rm B}(k) = \frac{1}{8\pi} \left\{ 1 + \frac{2T}{k} \ln \left[\frac{1 - \exp\big(-\frac{E_k + k}{2T}\big)}{1 - \exp\big(-\frac{E_k - k}{2T}\big)} \right] \right\} \;, \tag{228}$$

$$I_{\rm F}(k) = \frac{1}{8\pi} \left\{ 1 + \frac{2T}{k} \ln \left[\frac{1 + \exp\big(-\frac{E_k + k}{2T}\big)}{1 + \exp\big(-\frac{E_k - k}{2T}\big)} \right] \right\} \;. \tag{229}$$

9.4 Boltzmann equation in expanding universe

To go further we embed the Boltzmann equation in an expanding cosmological background. This implies that the time derivative in eq. (224) is re-interpreted as a covariant derivative. We assume a homogeneous and isotropic metric,

$${\rm d}s^2 = {\rm d}t^2 - a^2(t)\, {\rm d}\vec{x}^{\,2} \;, \tag{230}$$

as well as the energy-momentum tensor of an ideal fluid,

$$T_\mu{}^\nu = {\rm diag}(e, -p, -p, -p) \;, \tag{231}$$

where e denotes the energy density and p the pressure. In the Einstein equations, $G_\mu{}^\nu = 8\pi\, G\, T_\mu{}^\nu$, we denote

$$\frac{1}{m_{\rm Pl}^2} \equiv G \;, \tag{232}$$

where $m_{\rm Pl} \approx 1.22 \times 10^{19}$ GeV is the Planck mass. Then the "Hubble parameter" $H(t)$ satisfies

$$H(t) \equiv \frac{\dot{a}(t)}{a(t)} = \sqrt{\frac{8\pi}{3}} \frac{\sqrt{e(T)}}{m_{\rm Pl}} \; . \tag{233}$$

Furthermore, it can be shown that the age of the Universe and the temperature are related by

$$\frac{{\rm d}T}{{\rm d}t} = -T H(T) \big[3 c_s^2(T) \big] \; , \tag{234}$$

where $c_s^2 = \partial p / \partial e$ is the speed of sound squared.

With this setup, the Boltzmann equation takes the form

$$\left(\frac{\partial}{\partial t} - H k \frac{\partial}{\partial k} \right) f_k(t) = R(k) \; , \tag{235}$$

where R is the right-hand side of eq. (224).

Exercise 9.10. Show that the partial differential equation in eq. (235) can be re-expressed as an ordinary differential equation, if we move along a particular "trajectory" in the (k, t)-plane:

$$\frac{{\rm d}}{{\rm d}t} f_{k(t_0) \frac{a(t_0)}{a(t)}}(t) = R\left(k(t_0) \frac{a(t_0)}{a(t)} \right) \; . \tag{236}$$

This is called the "method of characteristics". Illustrate the procedure with a sketch.

In the end, it is convenient to measure time in terms of temperature rather than t, by making use of eq. (234). Let us furthermore approximate

$$3 c_s^2(T) \approx 1 \; , \quad \frac{a(t_0)}{a(t)} \approx \frac{T}{T_0} \; , \tag{237}$$

and write $k(t_0) \equiv \kappa T_0$. Then eqs. (224), (225), (234) and (236) can be combined into

$$T \frac{{\rm d} f_{\kappa T}}{{\rm d}T} \approx \frac{f_{\kappa T} - n_{\rm B}(E_{\kappa T})}{H(T) E_{\kappa T}} \Big\{ g_\psi^2 M^2 I_{\rm F}(\kappa T) + 2 g_\gamma^2 M^4 I_{\rm B}(\kappa T) \Big\} \; . \tag{238}$$

If we furthermore assume that the S-particles reach kinetic equilibrium, we may take the ansatz

$$f_{\kappa T} \simeq \frac{n(T)}{n_{\rm eq}(T)} n_{\rm B}(E_{\kappa T}) \; , \quad n_{\rm eq}(T) \equiv \int \frac{{\rm d}^3 \vec{k}}{(2\pi)^3} n_{\rm B}(E_k) \; , \tag{239}$$

where now the number density $n(T)$ of the S-particles is the unknown variable. If $n(T) = n_{\rm eq}(T)$, the system is in *chemical equilibrium*.

Exercise 9.11. Show that after inserting the ansatz of eq. (239) into eq. (238) and integrating over momenta, the evolution equation takes the form

$$T^4 \frac{\mathrm{d}}{\mathrm{d}T}\left\{\frac{n(T)}{T^3}\right\} = \frac{n(T) - n_{\mathrm{eq}}(T)}{H(T)} \left\langle \frac{g_\psi^2 M^2 I_{\mathrm{F}}(k) + 2g_\gamma^2 M^4 I_{\mathrm{B}}(k)}{E_k} \right\rangle , \tag{240}$$

where

$$\langle ... \rangle \equiv \frac{\int \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3} \, (...) \, n_{\mathrm{B}}(E_k)}{\int \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3} \, n_{\mathrm{B}}(E_k)} \tag{241}$$

denotes a "momentum average".

Exercise 9.12. Let us introduce the variable $z \equiv M/T$. What are the large-z limits of n_{eq}, $\langle I_{\mathrm{F}}(k)/E_k \rangle$ and $\langle I_{\mathrm{B}}(k)/E_k \rangle$? [These are needed for a part of exercise 9.13.]

9.5 Numerical integration of a Boltzmann equation

We are now interested in understanding what kind of solutions eq. (240) possesses.

For dimensional reasons, we may write $g_\gamma = g_\psi/\Lambda$, where Λ has the dimension of mass. If $\Lambda \gg M$, the contribution from g_γ can be omitted in comparison with that from g_ψ, and in the following we work under this assumption. Furthermore, for estimating $H(T)$, we assume $e(T) = 10T^4$.

Exercise 9.13. The solution of eq. (240) depends on the value of the dimensionless combination $X \equiv g_\psi^2 m_{\mathrm{Pl}}/M$. Present a numerical solution of $n(T)/T^3$ in the range $z \in (0.1, 40)$ for the cases $X \in \{10^{-2}, 10^{-1}, 1, 10, 100\}$, assuming the initial condition that $n = 0$ at $z = 0.1$. At which value of z does n equal n_{eq} in these cases? Which cases can be said to reach chemical equilibrium? Present an analytic solution describing the late-time behaviour of $n(T)$ at $z \gg 1$. [Hint: n_{eq} can be omitted if $z \gg 1$ and $n \gg n_{\mathrm{eq}}$.]

[The points obtained from exercise 9.13 will be multiplied by a factor $N = 5$.]