9 Scatterings, decays and inverse decays in cosmology

The basic tools defined in the previous sections — matrix elements, decay rates, scattering cross sections — play a role not only in collider experiments but also in statistical physics as well as its applications to cosmology and relativistic heavy ion collision experiments. Here we introduce the concept of a Boltzmann equation, and apply it to a simple example.

9.1 Boltzmann equation in flat spacetime

In collider experiments, the initial and final states are assumed to consist of single particles, described mathematically by plane waves or Gaussian wave packets. In statistical physics, the system contains very many particles. In the "thermodynamic limit", implying that the volume (V) is sent to infinity, there are infinitely many particles present. Their statistical properties are described by a "phase space distribution function", $f(x,\vec{k})$. If the system is homogeneous, as is generally a very good approximation, we can assume that f is independent of the spatial position $\vec{x}$. If the system is in full thermal equilibrium, f should also be independent of the time t. However it always depends on the momentum $\vec{k}$. The normalization is chosen such that the total number of particles, N, is given by

$$N(t) = \int_{\vec{x}\in V} \mathrm{d}^3\vec{x} \int \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3}\, f(x,\vec{k}) \; . \tag{217}$$

For a boson or fermion in thermal equilibrium, the distribution function reads

$$f_k \;\equiv\; f(x,\vec{k}) = \begin{cases} n_{\text{B}}(E_k) \equiv \dfrac{1}{\exp(E_k/T)-1} \; , & \text{for a boson} \\[2ex] n_{\text{F}}(E_k) \equiv \dfrac{1}{\exp(E_k/T)+1} \; , & \text{for a fermion} \end{cases} \; , \tag{218}$$

where T is the temperature (in natural units), n_{B} is the Bose distribution, n_{F} is the Fermi distribution, and $E_k = \sqrt{\vec{k}^2 + M^2}$. Below we denote the equilibrium value of f_k by f_{eq}.

At this point we need to modify the notation a bit from the vacuum case. From now on we denote $p \equiv |\vec{p}\,|$, whereas four-momenta are denoted with capital letters, $\mathcal{P} = (p^0, \vec{p}\,)$. The reason for the change is that the heat bath defines a special frame, so that expressions are not mainfestly Lorentz invariant, and energies and momenta often appear separately.

Denoting initial-state four-momenta by $\mathcal{K}$ and final-state four-momenta by $\mathcal{P}$, the phase space integration measure is defined similarly to eq. (208), except that there can now be $1+m$ particles in the initial state:

$$\int \mathrm{d}\Phi_{1+m\to n} \equiv \int \left\{ \prod_{a=1}^{m} \frac{\mathrm{d}^3\vec{k}_a}{(2\pi)^3 2E_{k_a}} \right\} \left\{ \prod_{i=1}^{n} \frac{\mathrm{d}^3\vec{p}_i}{(2\pi)^3 2E_{p_i}} \right\} (2\pi)^4 \delta^{(4)}\Big(\mathcal{K} + \sum_{a=1}^{m} \mathcal{K}_a - \sum_{i=1}^{n} \mathcal{P}_i \Big) \; . \tag{219}$$

Moreover, to avoid double counting in the phase space integrals, we introduce a statistical factor $c = \frac{1}{m_i! n_i!}$, where m_i, n_i are the numbers of identical particles in the initial and final states. Then the Boltzmann equation has the form

$$\mathcal{K}^\alpha \frac{\partial f_k}{\partial x^\alpha} = -\frac{c}{2} \sum_{m,n} \int \mathrm{d}\Phi_{1+m\to n}$$

$$\times \left\{ |\mathcal{M}|^2_{1+m\to n}\, f_k f_a \cdots f_m (1 \pm f_i) \cdots (1 \pm f_n) \right.$$

$$\left. - |\mathcal{M}|^2_{n\to 1+m}\, f_i \cdots f_n (1 \pm f_k)(1 \pm f_a) \cdots (1 \pm f_m) \right\} \,, \quad (220)$$

where $+$ applies to bosons, $-$ to fermions, $f_a, ..., f_m$ enumerate initial-state particles, and $f_i, ..., f_n$ final-state ones. The factors $1 \pm f$ are called Bose enhancement and Pauli blocking factors. On the last row of eq. (220), inverse reactions ("gain terms") have been introduced, in order to guarantee detailed balance in the case that all distribution functions have their equilibrium forms.

Exercise 9.1. Verify that if $|\mathcal{M}|^2_{n\to 1+m} = |\mathcal{M}|^2_{1+m\to n}$, and if all particles are in equilibrium, i.e. if $f_i \to f_{\mathrm{eq},i}$, then the right-hand side of eq. (220) vanishes. To show this it is important to note that $\mathrm{d}\Phi_{1+m\to n}$ guarantees energy conservation.

Exercise 9.2. Consider eq. (220) in a situation where only particles of type "k" are present in the initial state, i.e. all other distribution functions vanish, and where f_k is independent of $\vec{x}$. Show that the decay rate Γ of eq. (207) can be identified as $-\partial_t f_k / f_k$.

Exercise 9.3. Verify similarly that the Boltzmann equation is consistent with the scattering cross section as it was introduced in eqs. (202) and (203). For this, make use of $F \simeq 2E_A 2E_B |\vec{v}_A - \vec{v}_B|$ as well as $\rho_B(x) \simeq \int \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3} f_B(x, \vec{k})$.

9.2 Computation of matrix elements squared

We now assume that the world is described by QED with a single flavor of fermions, which we treat as massless, and a heavy scalar particle S of mass M, interacting as in eq. (184), *viz.*

$$\mathcal{L}_{\mathrm{QED}+S} = \mathcal{L}_{\mathrm{QED}} + \frac{1}{2}(\partial_\mu S)(\partial^\mu S) - \frac{M^2}{2} S^2 + g_\psi S\, \bar{\psi}\psi - g_\gamma\, S F_{\mu\nu} F^{\mu\nu} \,. \quad (221)$$

The Feynman rules for this theory were derived in sec. 7.4.3.

Exercise 9.4. Possible "tree-level" processes are $S \leftrightarrow e^- e^+$, $S \leftrightarrow \gamma\gamma$, $e^\pm \leftrightarrow S e^\pm$, $\gamma \leftrightarrow S\gamma$. Show that the last two are excluded by phase-space considerations (i.e. energy-momentum conservation).

Exercise 9.5. Show that $\sum |\mathcal{M}|^2$ for the decay $S \to e^-(p_1) e^+(p_2)$, summing over the spins of the final state particles, and $\sum |\mathcal{M}|^2$ for the "inverse decay" $e^-(k_a) e^+(k_b) \to S$, summing over the spins of the initial state particles, both equal $2 g_\psi^2 M^2$.

Exercise 9.6. Show that $\sum |\mathcal{M}|^2$ for the decay $S \to \gamma(p_1)\gamma(p_2)$, summing over the polarizations of the final state photons, and $\sum |\mathcal{M}|^2$ for the inverse decay $\gamma(k_a)\gamma(k_b) \to S$, summing over the polarizations of the initial photons, both equal $8g_\gamma^2 M^4$. To get the sum over polarizations, one can use a substitution inspired by eq. (79),

$$\sum_\lambda \epsilon_\mu^*(\vec{p}_1, \lambda)\epsilon_\nu(\vec{p}_1, \lambda) \to -g_{\mu\nu} \,. \tag{222}$$

Exercise 9.7. Note that eq. (222) amounts to summing both over physical and unphysical polarizations of photons (with one polarization state contributing with a minus sign), so the substitution looks problematic. However, the contribution from the unphysical states cancels. The reason is a "Ward identity", the statement that the amplitudes vanish when one substitutes $\epsilon_\mu(\vec{p}, \lambda) \to \mathcal{P}_\mu$ for any of the polarization vectors involved in a given process. Can you verify this explicitly in the above computation?

We now go on with eq. (220). Including a factor $c = 1/2$ for equivalent photons, and assuming spin equilibrium, i.e. that the distributions are independent of spin and polarization states, as well as translational invariance, the Boltzmann equation takes the form

$$\begin{aligned} \dot{f}_k = & -\frac{1}{2E_k} \int \frac{\mathrm{d}^3\vec{p}_1}{(2\pi)^3 2E_1} \frac{\mathrm{d}^3\vec{p}_1}{(2\pi)^3 2E_1} (2\pi)^4 \delta^{(4)}\big(\mathcal{K} - \mathcal{P}_1 - \mathcal{P}_2\big) \\ & \times \Big\{ 2g_\psi^2 M^2 \big[f_k(1 - f_1)(1 - f_2) - f_1 f_2 (1 + f_k) \big] \\ & + 4g_\gamma^2 M^4 \big[f_k(1 + f_1)(1 + f_2) - f_1 f_2 (1 + f_k) \big] \Big\} \,. \end{aligned} \tag{223}$$

Exercise 9.8. Let us assume that QED particles are in *kinetic equilibrium*, i.e. have Bose or Fermi distributions. Recalling also that they are massless, so that $E_i = p_i$, verify that in this case the Boltzmann equation takes the form

$$\begin{aligned} \dot{f}_k = & -\frac{f_k - n_\mathrm{B}(E_k)}{E_k} \int \frac{\mathrm{d}^3\vec{p}_1}{(2\pi)^3 2p_1} \frac{\mathrm{d}^3\vec{p}_2}{(2\pi)^3 2p_2} (2\pi)^4 \delta^{(4)}\big(\mathcal{K} - \mathcal{P}_1 - \mathcal{P}_2\big) \\ & \times \Big\{ g_\psi^2 M^2 \big[1 - n_\mathrm{F}(p_1) - n_\mathrm{F}(p_2) \big] + 2g_\gamma^2 M^4 \big[1 + n_\mathrm{B}(p_1) + n_\mathrm{B}(p_2) \big] \Big\} \,. \end{aligned} \tag{224}$$