8.3 Phase space integrals

Both the scattering amplitude and the particle decay require that one evaluates phase-space integrals, i.e. integrals of the form

$$\mathrm{d}\Phi_n(q \to \{p_f\}) = \prod_{i=1}^{n} \frac{\mathrm{d}^3\vec{p}_i}{(2\pi)^3 2E_i} (2\pi)^4 \delta^{(4)}(q - \sum_{i=1}^{n} p_i) . \tag{208}$$

To compute these, one eliminates some integrals using momentum conservation. The remaining ones are parameterized in terms of suitable variables, for example energies and angles.

Exercise 8.7. Show that after integrating over selected variables in order to remove δ-functions, the two-body phase space takes the form

$$\mathrm{d}\Phi_2(q \to p_1, p_2) \to \frac{\mathrm{d}\cos\theta\, \mathrm{d}\phi}{(2\pi)^2} \frac{|\vec{p}_1|}{4\sqrt{s}} , \quad s = q^2 . \tag{209}$$

The phase-space is Lorentz invariant, but the angles and the three momentum refer to the *rest frame* of the decaying particle. The angles θ and ϕ are relative to some fixed axis in the rest frame of q. For a scattering process $q = q_A + q_B$ and the rest frame of q is the center-of-mass frame and θ is chosen as the angle between $\vec{q}_A$ and $\vec{p}_1$.

Exercise 8.8. Show that the three momentum is given by

$$|\vec{p}_1| = |\vec{p}_2| = \frac{1}{2\sqrt{s}} \sqrt{\lambda(s, p_1^2, p_2^2)} , \tag{210}$$

where $\lambda(a, b, c) = (a - b - c)^2 - 4bc$. For the scattering $e^+e^- \to \mu^+\mu^-$, we have $p_1^2 = p_2^2 = m_\mu^2$, but derive the formula for the general case of unequal masses.

Exercise 8.9. Using this result, show that the $2 \to 2$ scattering cross section (203), evaluated in the center-of-mass frame, simplifies to

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega_{\text{c.m.}}} = \frac{1}{F} \frac{|\vec{p}_1|}{16\pi^2 E_{\text{c.m.}}} |\mathcal{M}(q_A, q_B \to p_1, p_2)|^2 , \tag{211}$$

where $\Omega_{\text{c.m.}}$ is the solid angle of particle 1 and $E_{\text{c.m.}} = E_A + E_B$ in this frame. Since the cross section does not depend on the azimuthal angle, we can write $\mathrm{d}\Omega_{\text{c.m.}} = 2\pi \sin\theta_{\text{c.m.}} \mathrm{d}\theta_{\text{c.m.}}$, where $\theta_{\text{c.m.}}$ is the scattering angle in the center of mass frame.

Exercise 8.10. Compute the total cross section for $2 \to 2$ scattering in ϕ^4 theory in the center-of-mass frame at a given center-of-mass energy.

As a final exercise, we evaluate the cross section following from eq. (201), *viz.* $\frac{1}{4}\sum|\mathcal{M}|^2 = 2e^4\,\frac{t^2+u^2}{s^2}$, in the center-of-mass frame in the high-energy limit, where one can neglect the electron and muon masses. We choose to parameterize the momenta as

$$q_A = E(1,0,0,1)\,, \qquad q_B = E(1,0,0,-1)\,, \tag{212}$$

$$p_1 = E(1,\sin\theta,0,\cos\theta)\,, \qquad p_2 = E(1,-\sin\theta,0,-\cos\theta)\,. \tag{213}$$

Exercise 8.11. Show that the differential muon production cross section is

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\alpha_{\rm em}^2}{4s}\,(1+\cos^2\theta)\,, \quad \alpha_{\rm em} \equiv \frac{e^2}{4\pi}\,, \tag{214}$$

and sketch the physical meaning of this result.

Exercise 8.12. Show that the total cross section reads

$$\sigma = \frac{4\pi\alpha_{\rm em}^2}{3s}\,. \tag{215}$$

For completeness, let us remark that the corresponding result with masses included reads

$$\sigma = \frac{4\pi\alpha_{\rm em}^2}{3s}\left(\frac{s-4m_\mu^2}{s-4m_e^2}\right)^{\frac{1}{2}}\left(1+\frac{2m_e^2}{s}\right)\left(1+\frac{2m_\mu^2}{s}\right)\theta(\sqrt{s}-2m_\mu)\,. \tag{216}$$