8 Computation of scattering cross sections

8.1 Reduction formula and S-matrix

In section 7.2 we have discussed how to obtain time-ordered vacuum correlation functions in perturbation theory. The scattering amplitudes can be extracted from these using the Lehmann-Symanzik-Zimmermann (LSZ) reduction, but a thorough derivation of this relation is beyond the scope of these exercises. The corresponding formalism is discussed in many QFT text books (even if a transparent general proof is difficult to locate).

Let us briefly sketch how the reduction formula is obtained by considering the four-point correlation function. To extract the contribution of an outgoing particle generated with field $\phi(x_1)$ one considers the Fourier transform with respect to x_1,

$$F(p_1, x_2, x_3, x_4) = \int \mathrm{d}^4x_1\, e^{ip_1\cdot x_1} \langle\Omega|\, T\left\{\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\right\}\, |\Omega\rangle\,. \tag{186}$$

One can show (and this is the subtle part) that if there is a single particle state with mass m, then for $p_1^0 > 0$ the function F develops a pole at $p_1^2 = m^2$, which describes the propagation of an outgoing particle at very large times,

$$F(p_1, x_2, x_3, x_4) = \frac{i}{p_1^2 - m^2} \langle\Omega|\phi(0)|\vec{p}_1; \text{out}\rangle\, \langle\vec{p}_1; \text{out}|\, T\left\{\phi(x_2)\phi(x_3)\phi(x_4)\right\}\, |\Omega\rangle + \ldots \tag{187}$$

The dots represent contributions from other states, which do not give rise to a pole. Now, the Feynman diagrams in momentum space do involve poles coming from the external propagators. The expectation value

$$\langle\Omega|\phi(0)|\vec{p}_1\rangle \;\equiv\; Z^{\frac{1}{2}} = 1 + O(\lambda) \tag{188}$$

is called the on-shell wave-function renormalization constant. It can be computed from the two-point function for which

$$F(p_1, x_2) = \frac{i}{p_1^2 - m^2} \langle\Omega|\phi(0)|\vec{p}_1\rangle\, \langle\vec{p}_1|\phi(x_2)\,|\Omega\rangle + \cdots = \frac{iZ}{p_1^2 - m^2}\, e^{ip_1\cdot x_2}\,. \tag{189}$$

We can repeat the procedure for the other three fields to get

$$\begin{aligned}\int \mathrm{d}^4x_1\, e^{ip_1\cdot x_1} \int \mathrm{d}^4x_2\, e^{ip_2\cdot x_2} \int \mathrm{d}^4x_3\, e^{-iq_A\cdot x_3} \int \mathrm{d}^4x_4\, e^{-iq_B\cdot x_4} \langle\Omega|\, T\left\{\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\right\}\, |\Omega\rangle \\ = \langle\vec{p}_1, \vec{p}_2; \text{out}|\vec{q}_A, \vec{q}_B; \text{in}\rangle\, \frac{i\sqrt{Z}}{p_1^2 - m^2}\, \frac{i\sqrt{Z}}{p_2^2 - m^2}\, \frac{i\sqrt{Z}}{q_A^2 - m^2}\, \frac{i\sqrt{Z}}{q_B^2 - m^2}\,.\end{aligned} \tag{190}$$

To get the $2 \to 2$ scattering amplitude one thus computes the Fourier transform of the four-point function and divides by the external propagators. A Green's function from which the external propagators have been removed is called an amputated Green's function. The quantity

$$S_{fi} = \langle\vec{p}_1, \vec{p}_2; \text{out}|\vec{q}_A, \vec{q}_B; \text{in}\rangle \tag{191}$$

is called the S-matrix. The non-trivial part of the S-matrix, in which the incoming particles interact, arises from the connected diagrams and is called the scattering matrix $\mathcal{M}$. It is defined as

$$(2\pi)^4\delta^{(4)}(q_A+q_B-p_1-p_2)\,i\mathcal{M}(q_A,q_B\to p_1,p_2)\;\equiv\;\langle\vec{p}_1,\vec{p}_2;\mathrm{out}|\vec{q}_A,\vec{q}_B;\mathrm{in}\rangle\big|_{\mathrm{connected}}\,. \tag{192}$$

In practical terms, the computation of $\mathcal{M}$ is quite simple. One uses the momentum-space Feynman rules for the connected Green's functions, removes the external propagators and multiplies by the appropriate Z-factor (at lowest order $Z=1$), and by $(-i)$. For ϕ^4-theory one obtains for the $2\to 2$ amplitude at the lowest order

$$\mathcal{M}=-\lambda\,. \tag{193}$$

For external fermions and photons, the reduction formula has the same structure, but one needs the matrix elements

$$\begin{aligned}\langle\Omega|\psi_\alpha(0)|e^-(\vec{p},s)\rangle &= Z_\psi^{\frac{1}{2}}u_\alpha(\vec{p},s)\,,\\ \langle\Omega|\bar{\psi}_\alpha(0)|e^+(\vec{p},s)\rangle &= Z_\psi^{\frac{1}{2}}\bar{v}_\alpha(\vec{p},s)\,,\\ \langle\Omega|A_\mu(0)|\gamma(\vec{p},\lambda)\rangle &= Z_A^{\frac{1}{2}}\epsilon_\mu(\vec{p},\lambda)\,,\end{aligned} \tag{194}$$

and their complex conjugates. Here u,v are Dirac spinors as determined in sec. 5.3, and ϵ_μ is a polarization vector from sec. 4.3.

Exercise 8.1. Verify the matrix elements in eq. (194) by using the expressions for the free field operators. They are obtained by replacing the Fourier expansion coefficients by creation and annihilation operators (for example, the relevant expression for a vector field is given by eq. (74)). Subsequently one uses canonical commutation relations, such as in eq. (175). The Z-factors for free fields are equal to 1.

One can formulate the above prescription as Feynman rules directly for $\mathcal{M}$. Suppressing the Z-factors, which for our leading-order computations are equal to 1, one first computes the amputated amplitude using the Feynman rules given in sec. 7. Then, for the external lines one multiplies with

1. 1 for in- or outgoing scalar fields,
2. $u(\vec{p},s)$ for an incoming fermion with spin s and $\bar{u}(\vec{p},s)$ for an outgoing fermion,
3. $\bar{v}(\vec{p},s)$ for an incoming anti-fermion and $v(\vec{p},s)$ for an outgoing anti-fermion,
4. $\epsilon(\vec{p},\lambda)$ for an incoming photon of polarization λ and $\epsilon^*(\vec{p},\lambda)$ for an outgoing photon.

One can also insert an overall $(-i)$, even though in the end it has no influence given that only $|\mathcal{M}|^2$ is physical.

Exercise 8.2. Show that the scattering amplitude $\mathcal{M}$ for the process $e^-(q_A, s_A)e^+(q_B, s_B) \to \mu^-(p_1, r_1)\mu^+(p_2, r_2)$ reads

$$\mathcal{M} = \frac{ie^2\, \bar{v}(\vec{q}_B, s_B)\gamma^\mu u(\vec{q}_A, s_A)\; \bar{u}(\vec{p}_1, r_1)\gamma_\mu v(\vec{p}_2, r_2)}{(q_A + q_B)^2}\;. \tag{195}$$

Only a single diagram contributes to this process, and the Feynman rules for the muon are exactly the same as for the electron.

Exercise 8.3. Compute the unpolarized amplitude squared, which is obtained by summing over the final state spins r_1, r_2 and averaging over the initial spins s_A, s_B:

$$\frac{1}{4}\sum_{s_A,s_B,r_1,r_2} |\mathcal{M}|^2 = \frac{1}{4}\sum_{s_A,s_B,r_1,r_2} \mathcal{M}^\dagger\, \mathcal{M}\;. \tag{196}$$

The conjugate Dirac structure follows from the relations ($\mu = 1\ldots4$)

$$\gamma^{\mu\dagger}\gamma^{0\dagger} = \gamma^0\gamma^\mu\,, \qquad\qquad \gamma^{5\dagger} = \gamma^5\,. \tag{197}$$

The sum over spins can be simplified using the completeness relations

$$\sum_s u(\vec{p}, s)\bar{u}(\vec{p}, s) = \not{p} + m \qquad\qquad \sum_s v(\vec{p}, s)\bar{v}(\vec{p}, s) = \not{p} - m\,. \tag{198}$$

(We are neglecting the mass in this exercise.) Show that the result reads

$$\frac{1}{4}\sum |\mathcal{M}|^2 = \frac{8e^4}{(q_A + q_B)^4}\big(q_A \cdot p_2\, q_B \cdot p_1 + q_A \cdot p_1\, q_B \cdot p_2\big)\;. \tag{199}$$

So-called Mandelstam variables are defined as $s = (q_A+q_B)^2$, $t = (q_A-p_1)^2$, $u = (q_A-p_2)^2$, where q_A, q_B are in-coming four-momenta, p_1, p_2 are out-going four-momenta, and $q_A + q_B = p_1 + p_2$ because of energy-momentum conservation.

Exercise 8.4. Show that the Mandelstam variables obey

$$s + t + u = q_A^2 + q_B^2 + p_1^2 + p_2^2\;. \tag{200}$$

Exercise 8.5. Show that in terms of Mandelstam variables, the result in eq. (199) can be expressed as

$$\frac{1}{4}\sum |\mathcal{M}|^2 = 2e^4\,\frac{t^2 + u^2}{s^2}\;. \tag{201}$$

8.2 Cross section and decay rate

With the scattering matrix $\mathcal{M}$ at hand, one can compute the probability P that two particles A and B scatter into a given final state. This probability is not just a property of the particles and their interactions. It is proportional the probability densities of the incoming particles and on their relative velocity. The cross section σ is defined by dividing out these factors:

$$\frac{\mathrm{d}P}{\mathrm{d}t\,\mathrm{d}^3\vec{x}} = |\vec{v}_A - \vec{v}_B|\,\rho_A(x)\rho_B(x)\,\sigma\,, \tag{202}$$

where it is assumed that the incoming particles both fly along the z-axis and $v_i = p_i^z/E_i$. The $2 \to n$ cross section is given by

$$\mathrm{d}\sigma = \frac{1}{F}\prod_{i=1}^{n}\frac{\mathrm{d}^3\vec{p}_i}{(2\pi)^3 2E_i}|\mathcal{M}(q_A, q_B \to \{p_f\})|^2 (2\pi)^4\delta^{(4)}\left(q_A + q_B - \sum_{i=1}^{n} p_i\right)\,, \tag{203}$$

where F is called the flux factor.

Exercise 8.6. If we place ourselves in the rest frame of particle B, we can replace $F \simeq 2E_A 2E_B|\vec{v}_A - \vec{v}_B|$ by the unambiguous $F = 4E_A m_B|\vec{v}_A|$. Show that this can be expressed in a Lorentz-invariant form as $F = 4\sqrt{(q_A\cdot q_B)^2 - m_A^2 m_B^2}$.

To obtain eq. (203), one computes the probability P for scattering normalized Gaussian wave packets of the form

$$|\phi(\vec{p}, L)\rangle \equiv \int \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3}\frac{1}{\sqrt{2E_k}}\phi(\vec{k}, \vec{p}, L)|\psi(\vec{k})\rangle \tag{204}$$

with

$$\phi(\vec{k}, \vec{p}, L) = \mathcal{N}\, e^{-(\vec{p}-\vec{k})^2 L^2} \tag{205}$$

and $|\vec{p}| \gg 1/L$. The normalization factor is chosen such that $\langle\phi(\vec{p}, L)\rangle|\phi(\vec{p}, L)\rangle = 1$. The particle density $\rho(x) = |\phi(x)|^2$ is obtained from the Fourier transform

$$\phi(x) = \int \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3}\phi(\vec{k}, \vec{p}, L)e^{-ik\cdot x}\,. \tag{206}$$

For the derivation one uses the fact that the wave packets are sharply peaked at $\vec{k} \approx \vec{p}$, so that one can replace the momenta $\vec{k}$ in the amplitudes by $\vec{p}$, the typical momentum of the wave packet, up to corrections of order $|\vec{p}| \gg 1/L$.

Apart from cross sections, we are interested in decay rates. In the limit where the decay width Γ is much smaller than the particle mass M, and we are in the rest frame of the decaying particle, its decay into a particular final state can be computed as

$$\mathrm{d}\Gamma = \frac{1}{2M}\prod_{i=1}^{n}\frac{\mathrm{d}^3\vec{p}_i}{(2\pi)^3 2E_i}|\mathcal{M}(q \to \{p_f\})|^2 (2\pi)^4\delta^{(4)}\left(q - \sum_{i=1}^{n} p_i\right)\,. \tag{207}$$

The total decay rate Γ_{tot} of an unstable particle is the sum of the rates into all possible decay channels and the lifetime is $\tau = 1/\Gamma_{\mathrm{tot}}$.