7.3 Wick's theorem

An important result for the evaluation of correlation functions in the free theory is Wick's theorem (cf. sec. 1). It states that

$$\langle 0| \boldsymbol{T} \{\phi(x_1)\ldots\phi(x_n)\} |0\rangle = \sum_{\text{pairings}} \Delta_F(x_{i_1}-x_{i_2})\Delta_F(x_{i_2}-x_{i_3})\ldots\Delta_F(x_{i_{n-1}}-x_{i_n}) \,, \quad (166)$$

where the Feynman propagator is defined as

$$\Delta_F(x-y) \equiv \langle 0| \boldsymbol{T}\{\phi(x)\phi(y)\} |0\rangle \,. \quad (167)$$

For a free theory, all higher-point correlators reduce to products of two-point correlation functions. The Feynman propagator has the Fourier representation

$$\Delta_F(x-y) = \int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{i}{k^2-m^2+i\varepsilon} e^{-ik\cdot(x-y)} \,, \quad \epsilon \equiv 0^+ \,. \quad (168)$$

There are different ways to derive Wick's theorem. One way is to use the path integral formalism, where a free field is represented by a set of high-dimensional Gaussian integrals for which we derived Wick's theorem in section 1. Let us briefly sketch how this is done. After adding a source term $b(x)\phi(x)$ to the Lagrangian, the Euclidean path integral for a scalar field has the form

$$\mathcal{Z}(b) = \int \mathcal{D}\phi(x)\, e^{-S_0^E[\phi]+\int \mathrm{d}^4x\, b(x)\phi(x)} \,. \quad (169)$$

Time-ordered products of fields are obtained as

$$\langle 0| T\{\phi(x_1)\ldots\phi(x_n)\} |0\rangle = \frac{1}{\mathcal{Z}(0)}\int \mathcal{D}\phi(x)\,\phi(x_1)\ldots\phi(x_n)\, e^{-S_0[\phi(x)]} \,. \quad (170)$$

One can get all these correlation functions by expanding in the source terms in the generating functional $\mathcal{Z}(b)$ as in eq. (11).

To make sense of eq. (169) one discretizes the theory on a lattice with points x_n. The field variables $\phi_n \equiv \phi(x_n)$ and corresponding source values $b_n \equiv b(x_n)$ play the same role as the coordinates x_n and the coefficients b_n in section 1. If the lattice is chosen to have finite volume and lattice spacing, it contains a finite number of points and associated field variables ϕ_n. The discretized Euclidean action with the source term then takes the form

$$S_E = \sum_{n,m} \hat{\phi}_n M_{nm} \hat{\phi}_m + \sum_n \hat{b}_n \hat{\phi}_n \,, \quad (171)$$

where we have rescaled the fields $\hat{\phi}_n$ and $\hat{b}_n$ to make them dimensionless. To obtain the matrix M_{nm} one has to discretize the derivatives in the action S_0^E. Computing $\mathcal{Z}(b)$ now reduces to evaluating (11) (with $x_n \to \hat{\phi}_n$) and we can immediately use Wick's theorem from eq. (18). After taking the continuum and infinite volume limits one then recovers eq. (166).

To derive eq. (166) in canonical quantization, one expresses the field operators in terms of creation and annihilation operators,

$$\phi(x) = \phi^+(x) + \phi^-(x) \,, \quad (172)$$

where

$$\phi^+(x) = \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{2\omega_p}\, a_p\, e^{-ip\cdot x} \qquad \text{and} \qquad \phi^-(x) = \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{2\omega_p}\, a_p^\dagger\, e^{ip\cdot x}\,. \tag{173}$$

Then one commutes all annihilation operators a_p to the right and all creation operators $a_p^\dagger$ to the left such that they annihilate the vacuum. This implies that the matrix elements in eq. (166) vanish up to commutator terms which arise during the rearrangement.

Exercise 7.9. Insert the decomposition in eq. (172) into eq. (167) and verify that

$$\Delta_F(x-y) = \begin{cases} \langle 0|[\phi^+(x), \phi^-(y)]|0\rangle & \text{for } x^0 > y^0 \\ \langle 0|[\phi^+(y), \phi^-(x)]|0\rangle & \text{for } y^0 > x^0 \end{cases} \tag{174}$$

Exercise 7.10. Derive the Fourier representation (168) for the commutator in eq. (174) using the commutation relations

$$[a_p, a_{p'}^\dagger] = (2\pi)^3(2\omega_p)\delta^{(3)}(\vec{p}-\vec{p}\,')\ , \quad [a_p, a_{p'}] = 0 = [a_p^\dagger, a_{p'}^\dagger] \tag{175}$$

for the creation and annihilation operators. To see that eq. (168) is equivalent to the expression obtained using the commutation relations it is simplest to first integrate over k^0 in eq. (168) by employing the residue theorem.

7.4 Feynman rules

7.4.1 ϕ^4-theory

Exercise 7.11. Compute the four-point correlation function

$$\langle\Omega|\,\boldsymbol{T}\,\{\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\}\,|\Omega\rangle \tag{176}$$

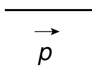
at first order in λ in ϕ^4-theory. The number of contractions is quite large, but many of them are equivalent because the fields in the interaction Lagrangian live at the same point. It is sufficient to write down the expression, you do not need to perform the integration over the position of the interaction vertex. Represent the contractions graphically by drawing the propagator $\Delta_F(x-y)$ as a line from x to y. Draw the diagrams for both the numerator in eq. (162) and also the normalization factor Z in eq. (163). Which contributions cancel against the normalization factor?

Exercise 7.12. Compute the Fourier transform of the connected part of the correlator (176). The connected piece is the most relevant, since it contains the scattering amplitude.

As a shortcut to Wick's theorem, the results for the correlation functions can be obtained using so-called Feynman rules. For the case of ϕ^4-theory, the momentum-space rules for the connected part of the n-point correlation function at m^{th} order in λ are as follows:

1. Draw all connected diagrams with n external legs and m interaction vertices. Then convert each diagram into the mathematical expression using the rules which follow.

2. Each line represents a propagator



$$= \frac{i}{p^2 - m^2 + i\epsilon} \tag{177}$$

3. At each vertex impose momentum conservation and insert a factor

$$= -i\lambda \tag{178}$$

4. Integrate over each momentum p flowing inside a closed loop: $\int \frac{\mathrm{d}^4 p}{(2\pi)^4}$.

5. Include an overall momentum conservation delta-function, $(2\pi)^4 \delta(P_{\text{tot}})$, where P_{tot} is the incoming minus the outgoing momentum.

6. For loop diagrams, divide by the appropriate symmetry factor (see below).

Let us comment on the symmetry factor. The interaction Lagrangian $-\frac{\lambda}{4!}\phi^4(x)$ involves a factor 4! which is absent in rule 3 because there are usually 4! possibilities to contract other fields to the vertex. Indeed, at tree level, the factor of 4! is always cancelled, but in loop diagrams, there are sometimes fewer possible contractions. Therefore, a division is left over.

7.4.2 QED

The Feynman rules for QED are similar to the scalar case. Wick's theorem applies also for vector and fermion fields (except that one has to be careful about signs with anti-commuting fermion fields), but we now need expressions for the fermion and photon propagators. In the so-called Feynman gauge the photon propagator reads

$$D_F^{\mu\nu}(x-y) = \langle 0| \boldsymbol{T}\big[A^\mu(x)\, A^\nu(y)\big] |0\rangle = \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{i}{k^2 + i\epsilon} (-g_{\mu\nu})\, e^{-ik\cdot(x-y)}\,. \tag{179}$$

The numerator $-g_{\mu\nu}$ arises from the sum over polarizations and includes unphysical degrees of freedom since $\partial_\mu D_F^{\mu\nu} \neq 0$, but one can show that the unphysical polarizations do not contribute to physical quantities. The photon propagator is not unique but depends on the gauge choice; furthermore, so-called Faddeev-Popov ghosts need to be introduced in general. The fermion propagator reads

$$S_{F\alpha\beta}(x-y) = \langle 0| \boldsymbol{T}\big[\psi_\alpha(x)\, \bar{\psi}_\beta(y)\big] |0\rangle = \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{i(k_\mu \gamma^\mu_{\alpha\beta} + m\delta_{\alpha\beta})}{k^2 - m^2 + i\epsilon}\, e^{-ik\cdot(x-y)}\,. \tag{180}$$

Typically, the abbreviation $\not{k} = k_\mu \gamma^\mu$ is used for the numerator and often one does not explicitly write out the Dirac indices α and β. Armed with these expressions and Wick's theorem, we can now evaluate correlation functions in QED.

Exercise 7.13. Compute the three-point function

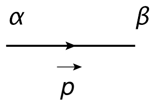
$$\langle\Omega|\, T\left\{\psi_\alpha(x_1)A_\mu(x_2)\bar{\psi}_\beta(x_3)\right\}\, |\Omega\rangle\,. \tag{181}$$

Represent the result graphically, using a line with an arrow to represent the fermion propagator and a wiggly line for the photon. Read off the Feynman rule for the QED vertex.

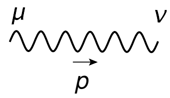
The QED Feynman rules in momentum space have the same structure as the ones in the scalar case, but the propagators and vertices now carry Dirac indices α, β and Lorentz indices μ, ν. They read as follows:

1. Draw all connected diagrams with n external legs and m interaction vertices. Then convert each diagram into the mathematical expression for the corresponding contraction of fields using the rules which follow.

2. Each line represents a propagator

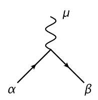


$$= \frac{i(\not{p}+m)_{\beta\alpha}}{p^2-m^2+i\epsilon}$$



$$= \frac{-ig_{\mu\nu}}{p^2+i\epsilon} \tag{182}$$

3. The vertex is



$$= -ie(\gamma^\mu)_{\beta\alpha} \tag{183}$$

4. Integrate over each momentum p flowing inside a closed loop: $\int\frac{d^4p}{(2\pi)^4}$.

5. Include an overall momentum conservation delta-function, $(2\pi)^4\delta^{(4)}(P_{\text{tot}})$, where P_{tot} is the incoming minus the outgoing momentum.

6. For loop diagrams: divide by the appropriate symmetry factor and include a factor (-1) for each closed fermion loop.

7.4.3 QED with an extra neutral scalar S

The lowest dimensional interaction terms of a neutral scalar S are

$$\mathcal{L}_{\text{int}} = g_\psi S(x)\bar{\psi}(x)\,\psi(x) - g_\gamma S(x)F_{\mu\nu}(x)F^{\mu\nu}(x)\,. \tag{184}$$

Exercise 7.14. Read off the Feynman rules associated with the vertices following from eq. (184). To get the Feynman rules in momentum space, it is easiest to write the action $S_{\text{int}} = \int d^4x\,\mathcal{L}_{\text{int}}$ in momentum space by Fourier transforming the fields using

$$S(x) = \int\frac{d^4k}{(2\pi)^4}\,\tilde{S}(k)\,e^{-ik\cdot x}\,, \tag{185}$$

and similarly for the other fields. Performing the integral over x yields the momentum conservation δ-function at the vertex, and the remaining expression yields the Feynman rules. To make the Feynman rule for the $S\gamma\gamma$ vertex user-friendly, one should symmetrize it in the indices and integration variables of the two photon fields.