7 Interacting fields in perturbation theory

7.1 Relativistically invariant interaction Lagrangians

Three examples of Lagrangian densities describing interacting fields are (ϕ is a real scalar, Φ a complex scalar field)

$$\mathcal{L}_{\phi^4} = \frac{1}{2}\partial_\mu \phi(x)\partial^\mu \phi(x) - \frac{m^2}{2}\phi^2(x) - \frac{\lambda}{4!}\,\phi^4(x)\,, \tag{153}$$

$$\mathcal{L}_{\text{QED}} = -\frac{1}{4}F^{\mu\nu}\,F_{\mu\nu} + \bar{\psi}(x)\,(i\not{D} - m)\,\psi(x)\,, \tag{154}$$

$$\mathcal{L}_{\text{Higgs}} = -\frac{1}{4}F^{\mu\nu}\,F_{\mu\nu} + (D_\mu\Phi)^*(D^\mu\Phi) + \mu^2\Phi^*\Phi - \frac{\lambda}{2}(\Phi^*\Phi)^2\,, \tag{155}$$

where $D_\mu = \partial_\mu + ieA_\mu$. The first Lagrangian describes a scalar field with self-interaction, the second one is Quantum Electrodynamics (QED) and describes a fermion interacting with the photon field. The third Lagrangian describes an electrically charged, complex scalar field which has both electromagnetic and self-interactions. It was used by P. Higgs in Phys. Rev. Lett. 13 (1964) 508 to illustrate how one can use a scalar field to generate a massive photon. Higgs's paper was initially rejected, because "we know" that the photon is massless, but the same mechanism now generates the masses of the Z and W bosons in the Standard Model.

All of these are local theories since the Lagrangian is obtained by an integral of a local function:

$$L(t) = \int \mathrm{d}^3\vec{x}\,\mathcal{L}(x)$$

For a given field content of the theory one can write down many additional local, Lorentz invariant terms in the Lagrangian. At first sight, it thus looks like there is infinite freedom in constructing local field theories. However, this freedom is strongly reduced by imposing *renormalizability*, which dictates that only terms with the mass dimension $D \leq 4$ are allowed in the Lagrangian. This condition arises once one considers quantum corrections, as will be explained in the QFT lectures.

While renormalizability restricts the operators in fundamental theories, higher-dimensional operators are allowed in low-energy effective theories. An effective theory is a quantum field theory which breaks down above a certain energy scale Λ (e.g. because there are additional particles with masses $M \sim \Lambda$ not included in the effective theory). In effective theories one includes higher-dimensional operators, but their contributions are suppressed by powers of Λ.

Exercise 7.1. In natural units $\hbar = c = 1$, the action S is a dimensionless quantity. Use this fact to derive the mass dimensions of the fields in the above Lagrangians.

Exercise 7.2. Consider a theory with only a single real scalar field $\phi(x)$ with a symmetry $\phi(x) \to -\phi(x)$. What is the most general Lagrangian density if terms up to dimension $D = 6$ are included? Which ones remain for $D \leq 4$?

Exercise 7.3. Consider QED, eq. (154). Again, extend this by writing down all terms up to mass dimension $D = 6$. All terms can be constructed from covariant derivatives D_μ and fermion fields $\psi(x)$. The QED Lagrangian is invariant under parity and one therefore does not need to write down pseudo-scalar terms such as $\bar{\psi}\gamma_5\psi$, which change sign under parity.

Exercise 7.4. Show that the Higgs Lagrangian in eq. (155) is invariant under a gauge transformation

$$\Phi(x) \to e^{i\alpha(x)}\,\Phi(x)\,, \qquad\qquad A_\mu(x) \to A_\mu(x) - \frac{\partial_\mu \alpha(x)}{e}\,. \tag{156}$$

Exercise 7.5. Note that the mass term in $\mathcal{L}_{\text{Higgs}}$ has the "wrong" sign. The Higgs potential $V = -\mu^2\Phi^*\Phi + \frac{\lambda}{2}(\Phi^*\Phi)^2$ has a minimum for a nonzero value of the field:

$$|\Phi(x)| = \phi_0 = \frac{v}{\sqrt{2}} = \left(\frac{\mu^2}{\lambda}\right)^{1/2}\,. \tag{157}$$

The quantity v is called the vacuum expectation value. Parameterize the complex Higgs field by two real fields $\beta(x)$ and $h(x)$ as

$$\Phi(x) = \frac{1}{\sqrt{2}} e^{i\beta(x)}\,(v + h(x))\,, \tag{158}$$

i.e. by a phase $\beta(x)$ and the radial deviation $h(x)$ from the vacuum value. Since the theory is invariant under the gauge transformation in eq. (156), we can perform a transformation which eliminates the field $\beta(x)$. This is called "unitary gauge". Write down $\mathcal{L}_{\text{Higgs}}$ in the unitary gauge. Show that the photon field A_μ picks up a mass term of the form

$$\Delta\mathcal{L} = \frac{1}{2} m_A^2 A_\mu A^\mu\,, \tag{159}$$

just like in eq. (72).

Some time ago, LHC data for the production of two photons showed tantalizing hints for the existence of an additional scalar particle S with a mass of $M_S = 750\,\text{GeV}$. With more data, it became clear that the hints were simply a statistical fluctuation, but it is an interesting exercise to try to write down a Lagrangian for a new scalar particle S. In the following we assume that the world is described by QED (with a single massive fermion, the electron e) and we add terms to the Lagrangian which describe the interaction of the photon and the fermion with the new scalar S. We take into account the following statements:

- Since the new particle arises in the decay $S \to \gamma\gamma$, it is electrically neutral and transforms trivially under a gauge transformation.
- Instead of a hadron collider such as the LHC, we consider e^+e^- collisions which then produce the scalar S.

- For the moment we assume that $S(x)$ is a scalar field (as opposed to a pseudoscalar). We also assume that our theory is parity invariant.
- The theory, and therefore also the new interactions with the scalar S, must be gauge invariant.

Exercise 7.6. Write down the lowest dimensional interaction terms of the fermion ψ with the new scalar S. Such terms should be present, otherwise S would not be directly produced in the e^+e^- collisions.

Exercise 7.7. Write down the lowest dimensional interaction terms of the scalar S with the photon field A_μ.

Exercise 7.8. Assume that S is a pseudoscalar, but that the theory is parity invariant. Write the lowest dimensional interactions in this case.

7.2 Time-dependent perturbation theory

Information on a quantum field theory can be extracted from the correlation functions of the fields. As shown by time-dependent perturbation theory in Quantum Mechanics (cf. discussion of the interaction / Dirac picture), of particular interest are time-ordered correlation functions

$$\langle\Omega|\,\boldsymbol{T}\left\{\phi(x_1)\ldots\phi(x_n)\right\}\,|\Omega\rangle\,. \tag{160}$$

The time-ordering operator $\boldsymbol{T}$ orders the fields such that the fields with largest time arguments are to the left. From the two-point function, one can obtain information about the spectrum of the theory, whereas the higher-point functions yield the scattering amplitudes. Interacting theories are in general too complicated to allow for an exact computation of these correlation functions, but we can simplify the problem by treating the interaction as a perturbation. For ϕ^4-theory, for example, we write the Lagrangian as

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{\text{int}} = \mathcal{L}_{\text{Klein-Gordon}} - \frac{\lambda}{4!}\phi^4(x) \tag{161}$$

and expand all observables as a power series in the coupling constant λ.

The prescription to compute correlation functions in perturbation theory is quite simple. One computes

$$\langle\Omega|\,\boldsymbol{T}\left\{\phi(x_1)\ldots\phi(x_n)\right\}\,|\Omega\rangle = \frac{1}{Z}\langle 0|\,T\left\{\phi(x_1)\ldots\phi(x_n)\,\exp\left[i\int \mathrm{d}^4z\,\mathcal{L}_{\text{int}}(z)\right]\right\}\,|0\rangle\,, \tag{162}$$

where the correlation function on the right-hand side is computed in the free theory. The normalization factor

$$Z = \langle 0|\,\boldsymbol{T}\left\{\exp\left[i\int \mathrm{d}^4z\,\mathcal{L}_{\text{int}}(z)\right]\right\}\,|0\rangle \tag{163}$$

arises because the vacuum of the free theory $|0\rangle$ is not the same as the vacuum of the interacting theory $|\Omega\rangle$. The formula (162) is derived in the QFT lectures. For perturbation theory, the factor

$$\exp\left[i\int \mathrm{d}^4 z\, \mathcal{L}_{\text{int}}(z)\right] = 1 + i\int \mathrm{d}^4 z\, \mathcal{L}_{\text{int}}(z) - \frac{1}{2}\int \mathrm{d}^4 z' \int \mathrm{d}^4 z\, \mathcal{L}_{\text{int}}(z)\mathcal{L}_{\text{int}}(z') + \dots \tag{164}$$

is expanded and the higher-order terms are suppressed by higher powers of the coupling constant. The interaction terms in the theory are polynomials in the fields and so the entire computation reduces to the evaluation of correlation functions in the free theory,

$$\langle 0|\, \boldsymbol{T}\{\phi(x_1)\dots\phi(x_n)\}\; |0\rangle\,, \tag{165}$$

where some of the fields arise from the expansion in eq. (164).