6 Non-relativistic limit of the Dirac equation

6.1 Dirac equation in the presence of an electromagnetic field

The Schrödinger equation for an electron in a background electromagnetic field can be obtained from the equation of the free particle

$$i\partial_t \chi = \frac{(i\vec{\nabla})^2}{2m}\chi \tag{134}$$

by performing the substitutions

$$i\partial_t \to iD_t = i\partial_t - e\phi\,, \qquad\qquad i\vec{\nabla} \to i\vec{D} = i\vec{\nabla} + e\vec{A}\,. \tag{135}$$

This gives the correct equation, except for the term describing the coupling of the electron spin to the magnetic field, which has the form

$$\Delta H = -\vec{\mu}\cdot\vec{B}\,, \tag{136}$$

where the magnetic moment is

$$\vec{\mu} = -g\mu_B\vec{S} = -g\frac{e}{2m}\frac{\vec{\sigma}}{2}\,. \tag{137}$$

The quantity $\mu_B = e/(2m)$ is called the Bohr magneton. The extra term ΔH, as well as the value of the so-called gyromagnetic factor g, which is equal to 2 to a very good approximation, will be derived from the Dirac equation in the next subsection. Historically, the value of g came as a surprise, since the energy associated with orbital angular momentum $\vec{L}$ of an electron in a magnetic field is $\Delta H = \mu_B\vec{L}\cdot\vec{B}$, without any extra factor.

In relativistic notation (remembering that $\partial^\mu = (\partial_t, -\vec{\nabla})$), the substitution in eq. (135) reads

$$i\partial_\mu \to iD_\mu = i\partial_\mu - eA_\mu\,, \tag{138}$$

and the quantity D_μ is called the covariant derivative. This suggests that the Dirac equation for an electron in an electromagnetic field is

$$(i\not{D} - m)\;\psi(x) = [\gamma_\mu(i\partial^\mu - eA^\mu) - m]\;\psi(x) = 0\,. \tag{139}$$

The deeper reason for the substitution (138) is that it leads to gauge invariance, i.e. invariance under the local phase symmetry

$$\psi(x) \to e^{i\alpha(x)}\psi(x)\,, \tag{140}$$

$$A_\mu(x) \to A_\mu(x) - \frac{\partial_\mu\alpha(x)}{e}\,, \tag{141}$$

where $\alpha(x)$ is an arbitrary, space-time dependent function. In the non-relativistic case, it is the field $\chi(x)$ which acquires the phase factor $\chi(x) \to e^{i\alpha(x)}\chi(x)$. Gauge symmetry plays a key role in particle physics and lies at the heart of the Standard Model.

Exercise 6.1. Show that after the substitution in eq. (135) the Schrödinger equation (134) and the Dirac equation (139) are gauge covariant.

6.2 Non-relativistic limit and the magnetic moment of the electron

We now consider the Dirac equation for a non-relativistic electron, for which the components of the spatial momentum are much smaller than the mass, i.e. $|\vec{p}| \ll m$. The energy

$$p^0 = \sqrt{m^2 + \vec{p}^2} = m + \frac{\vec{p}^2}{2m} + \ldots ,$$

on the other hand, is large because it includes the rest mass. To expand the Dirac equation in the non-relativistic limit, we first remove the large phase in the Dirac field, which arises due to the mass, and define

$$\psi(x) \equiv e^{-imt} \begin{pmatrix} \chi(x) \\ \eta(x) \end{pmatrix} , \tag{142}$$

where $\chi(x)$ and $\eta(x)$ are two-component fields. Having removed this factor, we can assume $D_\mu \chi(x) \ll m\, \chi(x)$ and $D_\mu \eta(x) \ll m\, \eta(x)$. For our discussion, it is furthermore convenient to use the so-called standard representation for the Dirac matrices,

$$\gamma^0 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} , \qquad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} . \tag{143}$$

This differs from the chiral representation in eq. (98). Both representations are commonly used. For massless particles, the chiral representation is especially convenient, while the standard representation is useful for studying the non-relativistic limit. As stressed earlier, all representations are equivalent, so that the choice is a matter of convenience.

Exercise 6.2. Derive the coupled equations for the fields $\chi(x)$ and $\eta(x)$. To keep the expressions compact, write everything in terms of covariant derivatives:

$$iD_0 \chi + i\vec{\sigma} \cdot \vec{D} \eta = 0 , \tag{144}$$

$$-(iD_0 + 2m)\eta - i\vec{\sigma} \cdot \vec{D} \chi = 0 . \tag{145}$$

Exercise 6.3. Show that eq. (145) implies that the field η is suppressed with respect to χ. For this reason η is called the “small” component of the Dirac field (this is true for particle solutions, for anti-particles the upper components are suppressed).

Exercise 6.4. Show that

$$(\vec{\sigma} \cdot \vec{a})\, (\vec{\sigma} \cdot \vec{b}) = \vec{a} \cdot \vec{b} + i\vec{\sigma} \cdot (\vec{a} \times \vec{b}) . \tag{146}$$

Exercise 6.5. Solve the lower-component equation for η and plug it into the upper-component equation for χ. This leads to the *Pauli equation* for an electron in an electromagnetic field,

$$iD_t\chi = \left[\frac{(i\vec{D})^2}{2m} - \frac{e}{2m}\,\vec{\sigma}\cdot\vec{B}\right]\chi\,. \tag{147}$$

One can now read off that $g = 2$.

The value of g changes because of quantum corrections associated with the electromagnetic field. Schwinger computed the first correction in 1948 and found $g = 2 + \frac{\alpha}{\pi} \approx 2(1 + 0.00116)$, where $\alpha = \frac{e^2}{4\pi} \approx 1/137$. This was in good agreement with the deviation from 2, which had been discovered experimentally shortly before. This agreement was a first triumph of the then new theory of Quantum Electrodynamics.

In modern experiments the anomalous magnetic moments $a = \frac{g-2}{2}$ of electrons and muons are measured with exquisite accuracy. The muon and electron magnetic moments differ because quantum corrections depend on the mass of the lepton. (The muon has the same electric charge as the electron but is about 200 times heavier.) The current experimental results are

$$a_e^{\rm exp} = (1\,159\,652\,180.73 \pm 0.28) \times 10^{-12}\,, \tag{148}$$

$$a_\mu^{\rm exp} = (1\,165\,920\,910 \pm 630) \times 10^{-12}\,. \tag{149}$$

The first, extremely precise number is used to obtain the value of α. The second number is compared to theoretical predictions and provides a test of the Standard Model of particle physics. The Standard Model theoretical prediction obtained by computing all quantum corrections up to order α^4 and even the most important of order α^5 is

$$a_\mu^{\rm SM} = (1\,165\,917\,760 \pm 662) \times 10^{-12}\,, \tag{150}$$

and deviates slightly from the experimental value.

Exercise 6.6. By how many standard deviations does the predicted theoretical value deviate from the measurement? What is the probability that this deviation is a statistical fluctuation, assuming that the distributions of both the experimental and theoretical values are Gaussian?

This deviation has been around since the experimental measurement in 2001. Despite a lot of scrutiny of the theory result (and the correction of some small mistakes), the discrepancy has persisted. Recently, a new more precise measurement has started, for more information see `http://muon-g-2.fnal.gov`. It will be interesting to see what happens!

6.3 Higher orders in the non-relativistic expansion

Above we derived the Pauli equation (i.e. Schrödinger equation with spin) by expanding the Dirac equation to first non-trivial order in $1/m$. Further new effects, such as terms describing the "fine structure" of the hydrogen spectrum, can be found by going to higher orders in an expansion in the inverse mass.

Exercise 6.7. In the measurement of the muon $g-2$, the muons are moving in an accelerator ring. They are fast enough that relativistic corrections to the Pauli equation (147) cannot be neglected. Compute the $1/m^2$ corrections to (147). They are obtained by including corrections to the lower-component equation for η before plugging it into the equation for χ. Note that up to terms suppressed as $1/m^3$, the correction term can be simplified using the leading-power equation for η. Using (147), the leading-power equation for χ, one can furthermore eliminate the time derivative in the $1/m^2$ terms. The $1/m^2$ correction to (147) consists of a spin-dependent and a spin-independent term. The spin-dependent term is called spin-orbit coupling and leads to the hydrogen fine structure. The spin-independent contribution is called the Darwin term.

Exercise 6.8. There is a subtlety concerning the $1/m^2$ correction terms derived in the previous exercise. Show that if one expands the charge in terms of the field χ, one finds

$$Q = \int \mathrm{d}^3\vec{x}\, j^0(x) = -e\int \mathrm{d}^3\vec{x}\, \bar{\psi}(x)\gamma^0\psi(x) = -e\int \mathrm{d}^3\vec{x}\, \chi^\dagger(x)\left[1 - \frac{1}{4m^2}(\sigma\cdot\vec{D})^2\right]\chi(x) + \ldots\,. \tag{151}$$

In quantum mechanics, the charge density of the electron field is just $-e$ times the probability density and so equation (151) makes it clear that in the presence of the $1/m^2$ correction term, the probability density is not $\chi^\dagger(x)\chi(x)$ as usual.

Exercise 6.9. Make a field redefinition

$$\chi(x) \to \left(1 + \frac{1}{8m^2}(\sigma\cdot\vec{D})^2\right)\chi(x)$$

to bring the probability density into the canonical form $\chi^\dagger(x)\chi(x)$. Show that in terms of the redefined field, the Pauli equation including the correction terms has the form

$$\left\{ iD_t + \frac{\vec{D}^2}{2m} + \frac{e\vec{\sigma}\cdot\vec{B}}{2m} + \frac{e(\vec{D}\cdot\vec{E} - \vec{E}\cdot\vec{D})}{8m^2} + \frac{ie\vec{\sigma}\cdot(\vec{D}\times\vec{E} - \vec{E}\times\vec{D})}{8m^2} \right\}\chi(x) = 0\,. \tag{152}$$