5.2 Dirac Lagrangian and the Dirac equation

From the considerations above it is clear that the action

$$S = \int \mathrm{d}^4x \, \bar{\psi}(x) \, (i\gamma^\mu \partial_\mu - m) \, \psi(x) \tag{114}$$

is a Lorentz scalar. Up to terms involving higher derivatives or γ_5, this action is unique. The factor i in the derivative term makes the action real.

Exercise 5.14. Show that S is real (ψ and $\bar{\psi}$ are assumed to vanish at infinity).

To get the equations of motion one can consider independent variations of the real and imaginary parts of the field $\psi(x)$. A shortcut leading to the same result is to consider independent variations of ψ and $\bar{\psi}$. The variation of $\bar{\psi}$ leads to the Dirac equation

$$(i\gamma^\mu \partial_\mu - m) \, \psi(x) = 0 \, . \tag{115}$$

Exercise 5.15. Multiply the Dirac equation by $(i\gamma^\mu \partial_\mu + m)$ from the left and show that each component of the spinor field fulfils the Klein-Gordon equation

$$\left(\Box + m^2\right) \psi(x) = 0 \, . \tag{116}$$

5.3 Plane-wave solutions of the Dirac equation

Similar to the Klein-Gordon equation, we now find plane-wave solutions to the Dirac equation. Let us first consider a plane wave of the form

$$\psi(x) = u(\vec{k}) \, e^{-ik\cdot x} \tag{117}$$

with $k^0 = \omega_k$ so that it fulfils the Klein-Gordon equation. The plane wave is a solution of the Dirac equation if the spinor $u(\vec{k})$ fulfils the matrix equation

$$(\not{k} - m) \, u(\vec{k}) = \begin{pmatrix} -m & k\cdot\sigma \\ k\cdot\bar{\sigma} & -m \end{pmatrix} u(\vec{k}) = 0 \, , \tag{118}$$

where we have inserted the chiral representation of the γ matrices and have introduced three common short-hand notations

$$\not{k} \equiv \gamma^\mu k_\mu \, , \qquad \sigma^\mu \equiv (\mathbb{1}, \sigma^i) \, , \qquad \bar{\sigma}^\mu \equiv (\mathbb{1}, -\sigma^i) \, . \tag{119}$$

The first one is called the Feynman slash.

As for the case of the Klein-Gordon equation, we also have "negative-frequency" solutions

$$\psi(x) = v(\vec{k})\, e^{+ik\cdot x} . \tag{120}$$

Exercise 5.16. Show that for these (which describe anti-particles) the spinor $v(\vec{k})$ fulfils the equation

$$(\not{k} + m)\, v(\vec{k}) = \begin{pmatrix} +m & k\cdot\sigma \\ k\cdot\bar{\sigma} & +m \end{pmatrix} v(\vec{k}) = 0 . \tag{121}$$

Consider the positive-frequency solutions with vanishing three-momentum $\vec{k} = \vec{0}$ and show that the spinor

$$u(0) = \sqrt{m} \begin{pmatrix} \xi \\ \xi \end{pmatrix} \tag{122}$$

fulfils the equation for an arbitrary two-component spinor ξ. According to eq. (102) the two-compoment spinor transforms under rotations as

$$\xi \to \exp(i\varphi_i \sigma^i/2)\, \xi . \tag{123}$$

This is the familiar transformation law of quantum mechanical two-component spinors and the spinor indeed describes two states of a spin-$\frac{1}{2}$ particle. The spinor $\xi = (1,0)^T$ fulfils $\sigma_3\, \xi = +1\, \xi$ and describes a particle with spin along the positive z direction, whereas $\xi = (0,1)^T$ has spin in the negative z direction.

Our work on the Lorentz transformations of the spinors now pays off. The solutions for particles with momentum $\vec{k}$ can be obtained by performing a boost of the solution for the particle at rest. We have derived the explicit form of the boost in eq. (104) and will now apply it to the spinor in eq. (122).

Exercise 5.17. Consider a boost along the third direction $\omega_{03} = -\omega_{30} = \beta$. Determine the value of β needed to arrive at a boost of the form

$$k^\mu = (m,0,0,0) \;\to\; k'^\mu = \Lambda^\mu{}_\nu k^\nu = (E,0,0,k^3) , \quad E = \omega_{k^3} = \sqrt{(k^3)^2 + m^2} . \tag{124}$$

Note that the explicit form of the boost matrix was given in eq. (96). (Since the other components do not play a role, it is convenient to write the vectors in the form $k^\mu = (\omega_{k^3}, k^3)$ to keep the notation compact.)

Exercise 5.18. Use the value of the boost parameter to boost the spinor $u(0)$ for the case $\xi = (1,0)^T$ so that it has three momentum $\vec{k} = (0,0,k^3)$. The relevant boost matrix is (104).

Exercise 5.19. The same exercise for the case $\xi = (0,1)^T$.

It turns out that the general boosted solution can be written in the elegant form

$$u(\vec{k}) = \begin{pmatrix} \sqrt{k\cdot\sigma}\,\xi \\ \sqrt{k\cdot\bar{\sigma}}\,\xi \end{pmatrix} , \tag{125}$$

where the square root of the matrix is obtained by first diagonalizing, taking the square root of the eigenvalues and transforming back.

Exercise 5.20. Derive the matrix relation

$$k \cdot \sigma \, k \cdot \bar{\sigma} = k^2 = m^2 \tag{126}$$

and use it to verify that eq. (125) indeed solves eq. (118).

Analogously, the general boosted solution for the anti-particle spinors can be written as

$$v(\vec{k}) = \begin{pmatrix} \sqrt{k \cdot \sigma}\, \eta \\ -\sqrt{k \cdot \bar{\sigma}}\, \eta \end{pmatrix} . \tag{127}$$

It is convenient to introduce orthonormal basis spinors ξ_s (e.g. $\xi_1 = (1,0)^T$ and $\xi_2 = (0,1)^T$) which fulfil

$$\xi_r^\dagger \xi_s = \delta_{rs} \,, \tag{128}$$

and then to denote the solution for the corresponding spinor by $u_s(\vec{k})$ where $s = 1, 2$. The two-component spinors then fulfil the completeness relation

$$\sum_{s=1,2} \xi_s \xi_s^\dagger = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} . \tag{129}$$

Exercise 5.21. Using the explicit result in eq. (125), show that the spinors are normalized as

$$u_s^\dagger(\vec{k})\, u_s(\vec{k}) = 2\omega_k \,, \tag{130}$$

$$\bar{u}_s(\vec{k})\, u_s(\vec{k}) = 2m \,. \tag{131}$$

Exercise 5.22. Using the results (125), (126) and (129) show that

$$\sum_{s=1,2} u_s(\vec{k})\, \bar{u}_s(\vec{k}) = \not{k} + m \,. \tag{132}$$

With a similar computation (which you are not asked to do), one finds

$$\sum_{s=1,2} v_s(\vec{k})\, \bar{v}_s(\vec{k}) = \not{k} - m \,. \tag{133}$$