5 Spinors and the Dirac equation

5.1 Lorentz transformation of spinor fields

To construct a theory for a field $\phi_\alpha(x)$, one first writes down an action. To get relativistic equations, this action must be Lorentz invariant. To construct such an action for a given field, it is obviously important to know how the field transforms under Lorentz transformations

$$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu\, x^\nu\,. \tag{85}$$

We know how Lorentz transformations act on a scalar field $\varphi(x)$ and on a vector field $A_\mu(x)$:

$$\varphi(x) \;\to\; \varphi'(x) = \varphi(\Lambda^{-1}x)\,, \tag{86}$$

$$A^\mu(x) \;\to\; A'^\mu(x) = \Lambda^\mu{}_\nu\, A^\nu(\Lambda^{-1}x)\,. \tag{87}$$

Exercise 5.1. Show that the term

$$\Delta\mathcal{L}(x) = \frac{m^2}{2} A_\mu(x) A^\mu(x)$$

in the Proca Lagrangian transforms as a scalar, i.e. $\Delta\mathcal{L}'(x) = \Delta\mathcal{L}(\Lambda^{-1}x)$. Remember that the metric tensor fulfils the identity

$$g_{\mu\nu}\Lambda^\mu{}_\rho\,\Lambda^\nu{}_\sigma = g_{\rho\sigma}\,. \tag{88}$$

Exercise 5.2. Show that the Lagrangian of a free massless scalar

$$\Delta\mathcal{L}(x) = \frac{1}{2}\partial_\mu\varphi(x)\,\partial^\mu\varphi(x)$$

transforms as a scalar.

The general transformation law for a field $\phi^\alpha(x)$ under Lorentz transformations is

$$\phi^\alpha(x) \;\to\; \phi^{\alpha\prime}(x) = D^\alpha{}_\beta(\Lambda)\,\phi^\beta(\Lambda^{-1}x)\,, \tag{89}$$

where the matrices $D(\Lambda)$ are a representation of the Lorentz group, i.e.

$$D(\Lambda_2)\,D(\Lambda_1) = D(\Lambda_2\Lambda_1)\,, \tag{90}$$

and $D(\mathbb{1}) = \mathbb{1}$. To find different Lorentz-invariant theories, one should now classify all possible representations of the Lorentz group. In the following, we will construct a representation for spin-$\frac{1}{2}$ particles. This is the most fundamental representation since higher spin representations can be obtained from products of spin-$\frac{1}{2}$ representations. Rather than analyzing full transformations, it is convenient to look at infinitesimal Lorentz transformations. One writes

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \Omega^\mu{}_\nu \tag{91}$$

where $\Omega^\mu{}_\nu$ is an infinitesimally small matrix.

Exercise 5.3. Show that the condition (88) implies that $\Omega^\mu{}_\nu$ is an antisymmetric matrix.

A general antisymmetric 4×4 matrix has six independent entries and can therefore be parameterized as

$$\Omega^\mu{}_\nu = -\frac{i}{2}\omega_{\alpha\beta}(J^{\alpha\beta})^\mu{}_\nu \,, \tag{92}$$

where the six antisymmetric matrices $J^{\alpha\beta}$ correspond to the six independent Lorentz transformations (3 rotations and 3 boosts), and the six parameters $\omega_{\alpha\beta}$ determine the angles of the rotations and the velocities of the boosts. The matrices $J^{\alpha\beta}$ are called the generators of the Lorentz transformations and have the form

$$(J^{\alpha\beta})^\mu{}_\nu = i(g^{\mu\alpha}\,\delta^\beta_\nu - g^{\mu\beta}\delta^\alpha_\nu)\,. \tag{93}$$

They fulfil the commutation relations in eq. (46),

$$\left[J^{\alpha\beta}, J^{\rho\sigma}\right] = i\left(g^{\beta\rho}J^{\alpha\sigma} - g^{\alpha\rho}J^{\beta\sigma} - g^{\beta\sigma}J^{\alpha\rho} + g^{\alpha\sigma}J^{\beta\rho}\right). \tag{94}$$

These commutation relations encode the Lorentz group in the same way that the commutation relations

$$\left[J^i, J^j\right] = i\epsilon^{ijk}J^k \tag{95}$$

describe the rotation group. It is convenient to analyze groups using the algebra of their generators and one can later reconstruct the finite transformations by exponentiation. A general Lorentz transformation can be written as

$$\Lambda = \exp\left(-\frac{i}{2}\omega_{\alpha\beta}J^{\alpha\beta}\right)\,. \tag{96}$$

Exercise 5.4. Consider $\omega_{12} = -\omega_{12} = \theta$ and all other components of $\omega_{\alpha\beta}$ zero. Show that the resulting transformation $\Lambda^\mu{}_\nu$ describes an infinitesimal rotation around the z-axis.

Exercise 5.5. Consider $\omega_{01} = -\omega_{10} = \beta$ and all other components zero. Show that the resulting transformation $\Lambda^\mu{}_\nu$ describes an infinitesimal boost along the x-axis.

In section 3 we considered the Dirac matrices and showed that the six matrices

$$S^{\alpha\beta} = \frac{i}{4}\left[\gamma^\alpha, \gamma^\beta\right] \tag{97}$$

fulfil the commutation relations (94) of the Lorentz group. To get an explicit form of these matrices, we need an explicit form of the Dirac matrices. We will use the so-called chiral (or Weyl) representation. Writing the 4×4 matrices in 2×2 blocks, the matrices have the form

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \tag{98}$$

where σ^i are the Pauli matrices.

Using the matrices in eq. (97), we can now construct the spinor representation of the Lorentz group. We consider a field ψ with four complex components, called a Dirac spinor, which transforms as

$$\psi(x) \;\to\; \psi'(x) = D(\Lambda)\psi(\Lambda^{-1}x) \;, \tag{99}$$

where

$$\Lambda = \exp\Big(-\frac{i}{2}\omega_{\alpha\beta}\, J^{\alpha\beta}\Big) \,, \tag{100}$$

$$D(\Lambda) = \exp\Big(-\frac{i}{2}\omega_{\alpha\beta}\, S^{\alpha\beta}\Big) \,. \tag{101}$$

Exercise 5.6. Consider a general rotation and write the rotation parameters as $\omega_{ij} = -\epsilon_{ijk}\varphi_k$. Show that the transformation takes the form

$$D(\Lambda) = \begin{pmatrix} \exp(i\varphi_i\sigma^i/2) & 0 \\ 0 & \exp(i\varphi_i\sigma^i/2) \end{pmatrix} \,. \tag{102}$$

Exercise 5.7. Consider a rotation around the z-axis and show that after a rotation with $\omega_{12} = -\varphi_3 = 2\pi$ one obtains the remarkable result

$$\psi(x) \;\to\; \psi'(x) = -\psi(x) \,. \tag{103}$$

Spinors pick up sign under a 2π rotation (while vectors rotate onto themselves)!

Exercise 5.8. Consider a boost and write the boost parameters as $\omega_{0i} = \beta_i$. Show that in the chiral representation the boost matrix takes the form

$$D(\Lambda) = \begin{pmatrix} \exp(-\beta_i\sigma^i/2) & 0 \\ 0 & \exp(\beta_i\sigma^i/2) \end{pmatrix} \,. \tag{104}$$

It is interesting to note that our representation matrices are block-diagonal. This means that the spinor representation is *reducible*. One can split the spinor into two-component spinors

$$\psi(x) = \begin{pmatrix} \psi_L(x) \\ \psi_R(x) \end{pmatrix} \,, \tag{105}$$

which transform independently. The left-handed and right-handed spinors $\psi_{L,R}(x)$ are called Weyl spinors and can be extracted from a general representation of the γ matrices using the projection operators

$$\psi_L(x) = P_L\, \psi(x) \,, \qquad\qquad \psi_R(x) = P_R\, \psi(x) \,, \tag{106}$$

where

$$P_L = \frac{1}{2}(\mathbb{1} - \gamma_5)\,, \qquad\qquad P_R = \frac{1}{2}(\mathbb{1} + \gamma_5)\,, \tag{107}$$

and, in the chiral representation,

$$\gamma^5 = \begin{pmatrix} -\mathbb{1} & 0 \\ 0 & +\mathbb{1} \end{pmatrix} . \tag{108}$$

The spinors are called left and right-handed, because they have definite helicity (projection of the spin on the momentum) in the massless case.

The boost matrix $D(\Lambda)$ in eq. (104) is not unitary. Because of this, $\psi^\dagger(x)\psi(x)$ does not transform as a scalar. To find a Lorentz-invariant quantity, note that the Dirac matrices in the chiral representation have the property

$$\gamma^0\gamma^\mu\gamma^0 = (\gamma^\mu)^\dagger . \tag{109}$$

Exercise 5.9. Show that this implies

$$(S^{\alpha\beta})^\dagger = \gamma^0 S^{\alpha\beta}\gamma^0 . \tag{110}$$

Exercise 5.10. After defining the adjoint spinor $\bar{\psi}(x) \equiv \psi^\dagger(x)\gamma^0$ one finds with eq. (110) that it transforms as

$$\bar{\psi}(x) \;\rightarrow\; \bar{\psi}'(x) = \bar{\psi}(\Lambda^{-1}x)\, D(\Lambda)^{-1}\,, \quad \text{i.e.} \qquad D^\dagger\gamma^0 = \gamma^0 D^{-1} . \tag{111}$$

It is easiest to show this using an infinitesimal transformation. This implies that the product $\bar{\psi}(x)\,\psi(x)$ transforms as a Lorentz scalar.

Exercise 5.11. Show that $\bar{\psi}(x)\gamma_5\psi(x)$ is a pseudoscalar, i.e. invariant in proper Lorentz-transformations but odd in "parity", defined as $\psi \rightarrow \gamma^0\psi$. For the Lorentz part, you may again make use of infinitesimal transformations.

Exercise 5.12. Show that

$$D(\Lambda)^{-1}\gamma^\mu D(\Lambda) = \Lambda^\mu{}_\nu\gamma^\nu . \tag{112}$$

It is easiest to verify this using an infinitesimal transformation and the relations in eqs. (47) and (93),

$$\left[S^{\alpha\beta}, \gamma^\mu\right] = i\gamma^\alpha g^{\beta\mu} - i\gamma^\beta g^{\alpha\mu} = -(J^{\alpha\beta})^\mu{}_\nu\gamma^\nu . \tag{113}$$

Exercise 5.13. Show that the relation (112) immediately implies that $\bar{\psi}(x)\gamma^\mu\psi(x)$ transforms as a Lorentz vector and $\bar{\psi}(x)\gamma^\mu\gamma^\nu\psi(x)$ as a tensor.