4 Klein-Gordon, Maxwell and Proca equations

4.1 Klein-Gordon equation

The relativistic wave equation for a free spin zero field of mass m has the form

$$\left(\Box + m^2\right) \varphi(x) = 0 \,, \quad \Box \equiv \partial_t^2 - \nabla^2 \,, \tag{64}$$

where $\varphi(x)$ is real or complex. This is called the Klein-Gordon equation.

Exercise 4.1. Derive the equality

$$\int \frac{\mathrm{d}^3\vec{k}}{2\omega_k}\, f(\vec{k}\,) = \int \mathrm{d}^4k\, \delta(k^2 - m^2)\, \theta(k^0)\, f(\vec{k}\,)\,, \tag{65}$$

where $\omega_k \equiv \sqrt{m^2 + |\vec{k}|^2}$. Show that the integration measure on the right-hand side is invariant under proper Lorentz transformations. The same is thus true for the one on the left-hand side.

Exercise 4.2. Show that the following general superposition of plane waves ($k \cdot x \equiv k^0x^0 - \vec{k} \cdot \vec{x}$)

$$\varphi(x) = \int \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3\, 2\omega_k} \left[a(\vec{k})\, e^{-ik\cdot x} + b(\vec{k})\, e^{ik\cdot x}\right] \,, \tag{66}$$

with $k^0 = \omega_k$, solves the Klein-Gordon equation.

Exercise 4.3. For a real field with $\varphi(x) = \varphi^*(x)$, the expansion coefficients $a(\vec{k})$ and $b(\vec{k})$ are related. What is their relation?

4.2 Electromagnetic field in relativistic notation

The Maxwell equations (in Lorentz-Heaviside units) are

$$\nabla \cdot \vec{E} = \rho \,, \qquad \nabla \times \vec{B} - \partial_t\vec{E} = \vec{j}\,, \tag{67}$$

$$\nabla \cdot \vec{B} = 0 \,, \qquad \nabla \times \vec{E} + \partial_t\vec{B} = 0 \,. \tag{68}$$

Exercise 4.4. Show that the homogeneous equations (68) are automatically satisfied if the fields are written in terms of the 4-vector potential $A^\mu = (\phi, A^i)$, with

$$\vec{A} = \sum_i A^i\vec{e}_i \,, \quad \vec{B} = \nabla \times \vec{A} \,, \quad \vec{E} = -\partial_t\vec{A} - \nabla\phi \,. \tag{69}$$

Exercise 4.5. Defining the field strength tensor

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \tag{70}$$

and current vector $j^\mu = (\rho, \vec{j})$, show that $E^i = -F^{0i}$ and $B^i = -\frac{1}{2}\epsilon^{ijk}F_{jk}$ and that the inhomogeneous equations (67) can be written as

$$\partial_\mu F^{\mu\nu} = j^\nu \, . \tag{71}$$

4.3 Proca equation

Exercise 4.6. Derive the equations of motion for the field A_μ from the Lagrangian density

$$\mathcal{L}(A) = -\frac{1}{4}F^{\mu\nu}\,F_{\mu\nu} + \frac{m^2}{2}A_\mu A^\mu - j^\mu A_\mu \, , \tag{72}$$

and show that they reduce to the Maxwell equations (71) for $m = 0$. The equation with $m \neq 0$ is called the Proca equation and describes the propagation of a massive spin-1 field.

Exercise 4.7. Show that the Proca equation implies the consistency relation $m^2\partial_\mu A^\mu = \partial_\mu j^\mu$. For a conserved current $\partial_\mu j^\mu = 0$ and non-zero mass m, this imposes the condition $\partial_\mu A^\mu = 0$. This condition implies that not all four components of the field A^μ are independent.

Exercise 4.8. Show that using $\partial_\mu A^\mu = 0$ the Proca equation simplifies to

$$\left(\Box + m^2\right) A_\mu(x) = j_\mu(x) \, . \tag{73}$$

In other words, for $j_\mu = 0$, each component of the Proca field fulfils the Klein-Gordon equation.

As in eq. (66), we can write the solution of eq. (73) as a superposition of plane waves. For a real field it is often written in the form

$$A_\mu(x) = \sum_{\lambda=0}^{3} \int \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3\, 2\omega_k} \left[\epsilon_\mu(\vec{k},\lambda)a(\vec{k},\lambda)e^{-ik\cdot x} + \epsilon^*_\mu(\vec{k},\lambda)a^*(\vec{k},\lambda)e^{ik\cdot x}\right] , \tag{74}$$

where the four auxiliary *polarization vectors* $\epsilon_\mu(\vec{k},\lambda)$ can be chosen such that they form an orthonormal basis of Minkowski space-time:

$$\epsilon^*_\mu(\vec{k},\lambda)\epsilon^\mu(\vec{k},\lambda') = g_{\lambda\lambda'} \, . \tag{75}$$

In principle, these vectors can be chosen real, but sometimes it is convenient to work with complex ones (circular polarizations). The first two of the vectors (the transverse polarizations) are usually chosen to have the form

$$\epsilon^\mu(\vec{k},1) = (0, \vec{\epsilon}(\vec{k},1)) \, , \quad \epsilon^\mu(\vec{k},2) = (0, \vec{\epsilon}(\vec{k},2)) \, , \tag{76}$$

with $k \cdot \epsilon(\vec{k},1) = k \cdot \epsilon(\vec{k},2) = 0$ and $\vec{\epsilon}^*(\vec{k},i) \cdot \vec{\epsilon}(\vec{k},j) = \delta_{ij}$. The third space-like polarization vector is chosen to have its three-vector parallel to $\vec{k}$. This longitudinal polarization vector has the form

$$\epsilon^\mu(\vec{k},3) = (A, B\,\vec{k}) \, , \tag{77}$$

where the parameters A and B are chosen such that $\epsilon^\mu(\vec{k},3)$ is orthogonal to k, $k \cdot \epsilon(\vec{k},3) = 0$, and normalized according to eq. (75). The final polarization vector points along the direction of k^μ:

$$\epsilon^\mu(\vec{k},0) = C\, k^\mu\,. \tag{78}$$

Exercise 4.9. Determine the coefficients A, B and C and show that the consistency condition $\partial_\mu A^\mu = 0$ implies that $a(\vec{k},0) = 0$. The massive field thus contains three independent solutions ("polarizations") for a given momentum.

Exercise 4.10. Show that the four polarization vectors fulfil the completeness relation

$$\sum_{\lambda,\lambda'=0}^{3} g_{\lambda\lambda'}\, \epsilon^*_\mu(\vec{k},\lambda)\epsilon_\nu(\vec{k},\lambda) = g_{\mu\nu}\,. \tag{79}$$

Exercise 4.11. Derive from this that the three physical polarization vectors fulfil the completeness relation

$$\sum_{\lambda=1}^{3} \epsilon^*_\mu(\vec{k},\lambda)\epsilon_\nu(\vec{k},\lambda) = -g_{\mu\nu} + \frac{k_\mu k_\nu}{m^2}\,, \tag{80}$$

which follows immediately from rewriting the previous relation (79).

Please note the factor of $1/m^2$, which makes the limit $m \to 0$ nontrivial.

4.4 Gauge invariance

Exercise 4.12. Show that for $m = 0$ and assuming current conservation the Lagrangian $\mathcal{L}(A)$ is invariant under gauge transformations

$$A_\mu(x) \to A_\mu(x) + \partial_\mu\, \alpha(x)\,, \tag{81}$$

where $\alpha(x)$ is an arbitrary differentiable field. These transformations leave $F^{\mu\nu}$ and therefore also the electric and magnetic fields unchanged.

Exercise 4.13. Show that for *any* given field A_μ one can always make a gauge transformation such that the condition $\partial_\mu A^\mu = 0$ is fulfilled. This is called Lorenz gauge (since the imposed condition is Lorentz invariant…).

Exercise 4.14. The condition $\partial_\mu A^\mu = 0$ eliminates one unphysical degree of freedom. However, even imposing $\partial_\mu A^\mu = 0$ does not fix the gauge freedom completely, since we can still perform gauge transformations which fulfil the scalar wave equation

$$\Box\, \alpha(x) = 0\,. \tag{82}$$

Such transformations correspond to another unphysical degree of freedom so that we conclude that the photon field only contains two physical degrees of freedom.

The Proca action for a massive spin-1 field is not gauge invariant, but the *Stueckelberg action* (see `http://www.stuckelberg.org`)

$$\mathcal{L}(A,\phi) = -\frac{1}{4}F^{\mu\nu}\,F_{\mu\nu} + \frac{m^2}{2}(\partial_\mu\phi + A_\mu)(\partial^\mu\phi + A^\mu) - j^\mu A_\mu\,, \tag{83}$$

which involves an additional real scalar field ϕ is invariant under the simultaneous gauge transformation

$$A_\mu(x) \to A_\mu(x) + \partial_\mu\,\alpha(x)\,. \qquad\qquad \phi(x) \to \phi(x) - \alpha(x)\,. \tag{84}$$

Choosing the gauge condition $\phi(x) = 0$, the Stueckelberg action reduces to the Proca action. One can thus view a massive spin-1 field as a combination of a gauge invariant field $A_\mu(x)$, with two degrees of freedom, and an additional scalar field ϕ, which provides the third degree of freedom (the longitudinal polarization). An explicit realization of this is the Higgs mechanism, where additional scalar fields provide a mass term for the gauge fields of the weak interactions.