2 Integrals over Grassmann variables

To construct a consistent quantum theory of *fermionic* fields, the field operators must fulfil the *anti-commutation* relations

$$\left\{\psi_\alpha(x), \psi^\dagger_\beta(y)\right\} = \delta^{(3)}(\vec{x} - \vec{y}\,)\, \delta_{\alpha\beta}\,, \tag{20}$$
$$\left\{\psi_\alpha(x), \psi_\beta(y)\right\} = \left\{\psi^\dagger_\alpha(x), \psi^\dagger_\beta(y)\right\} = 0\,,$$

at equal times $x^0 = y^0$. To construct a path-integral representation of such a theory, one uses *anti-commuting variables*, which are also called Grassmann numbers. They are introduced and explored in this section. We consider a set of n such numbers η_i, $i = 1 \dots n$, which fulfil the Grassmann algebra

$$\left\{\eta_i, \eta_j\right\} = 0\,, \tag{21}$$

implying that $\eta_i^2 = 0$, which makes the algebra extremely simple. The most general function of two Grassmann variables is

$$f(\eta_1, \eta_2) = f_{00} + f_{10}\,\eta_1 + f_{01}\,\eta_2 + f_{11}\,\eta_1\eta_2\,, \tag{22}$$

since $\eta_2\eta_1 = -\eta_1\eta_2$ and all higher-order terms vanish. The expansion coefficients are ordinary numbers. The Taylor expansion of Grassmann functions is thus always finite and exact. In the rest of this section, Latin letters refer to real or complex numbers, Greek letters denote the Grassmann variables.

Exercise 2.1. The Grassmann algebra can be implemented using anti-commuting matrices. Provide a matrix representation for the case $n = 2$. *Hint:* It is not necessary to use 2×2 matrices.

Exercise 2.2. Equation (22) shows that the $n = 2$ algebra is four-dimensional. What is the dimension for general n?

The integral over a Grassmann function $f(\eta) = a + b\,\eta$ is defined as

$$\int \mathrm{d}\eta\, f(\eta) \equiv b\,. \tag{23}$$

Exercise 2.3. Show that, up to a normalization factor, this definition follows from the requirements of linearity and invariance of the integral under a shift $\eta \to \eta + \theta$.

One can also define the derivative of a Grassmann function as

$$\frac{\partial}{\partial\eta} f(\eta) = \frac{\partial}{\partial\eta}\,(a + b\,\eta) \equiv b\,, \tag{24}$$

which happens to be the same as the integral. For multiple integrals and derivatives one needs to adopt a sign convention. We define

$$\int \mathrm{d}\eta_2 \int \mathrm{d}\eta_1\, \eta_1\eta_2 \equiv \frac{\partial}{\partial\eta_2}\,\frac{\partial}{\partial\eta_1}\,\eta_1\eta_2 \equiv +1\,, \tag{25}$$

i.e. we perform the innermost integral (or derivative) first.

Exercise 2.4. Compute the Gaussian integrals

$$I(A) = \int \mathrm{d}\eta_n \int \mathrm{d}\eta_{n-1} \dots \int \mathrm{d}\eta_1 \, e^{-\underline{\eta}^T A \, \underline{\eta}} \, , \tag{26}$$

where $\underline{\eta} = (\eta_1, \dots, \eta_n)^T$ for $n = 2, 3, 4$. *Hints:* Taylor expand and note that only the terms proportional to $\eta_1 \eta_2 \dots \eta_n$ contribute, which involve exactly one power of each variable. Note also that A can be chosen anti-symmetric.

Under a variable change $\theta = a\eta$, we have

$$1 = \int \mathrm{d}\theta \, \theta = \int \mathrm{d}(a\eta) \, a\eta \tag{27}$$
$$\Rightarrow \mathrm{d}(a\eta) = \frac{1}{a} \mathrm{d}\eta \, .$$

This is the opposite of the behaviour of regular (bosonic) integrals, where $\mathrm{d}(ax) = a \, \mathrm{d}x$. The Grassmann integral behaves like a derivative under variable transformations, which may not be surprising, since it is the same as the derivative.

Exercise 2.5. For a general variable transformation $\xi_i = B_{ij} \eta_j$, prove that

$$\mathrm{d}\xi_n \dots \mathrm{d}\xi_1 = (\det B)^{-1} \mathrm{d}\eta_n \dots \mathrm{d}\eta_1 \, . \tag{28}$$

Hint: proceed as in eq. (27), using $\eta_{i_1} \dots \eta_{i_n} = \epsilon_{i_1 \dots i_n} \eta_1 \dots \eta_n$.

To work with the complex-valued Dirac field, one introduces complex Grassmann variables

$$\eta \equiv \frac{1}{\sqrt{2}} (\eta_1 + i\eta_2) \ , \quad \eta^* \equiv \frac{1}{\sqrt{2}} (\eta_1 - i\eta_2) \ . \tag{29}$$

One can treat η and η^* as independent variables and define

$$\int \mathrm{d}\eta^* \mathrm{d}\eta \, \eta \, \eta^* \equiv 1 \, . \tag{30}$$

Exercise 2.6. Derive the following identities for Gaussian integrals with complex Grassmann variables:

$$\left(\prod_{i=1}^{n} \int \mathrm{d}\eta_i^* \int \mathrm{d}\eta_i \right) e^{-\underline{\eta}^\dagger A \, \underline{\eta}} = \det A \, , \tag{31}$$

$$\left(\prod_{i=1}^{n} \int \mathrm{d}\eta_i^* \int \mathrm{d}\eta_i \right) e^{-\underline{\eta}^\dagger A \, \underline{\eta} + \underline{\eta}^\dagger \underline{\theta} + \underline{\theta}^\dagger \underline{\eta}} = \det A \; e^{\underline{\theta}^\dagger A^{-1} \underline{\theta}} \, , \tag{32}$$

for a Hermitian matrix A. *Hint:* Derive eq. (31) by performing a change of variables which diagonalizes A, and complete the square to obtain eq. (32).

Except for the normalization, the only difference to regular (bosonic) Gaussian integrals is the appearance of $\det A$ instead of $(\det A)^{-1}$. As in the bosonic case, all moments can be obtained by taking derivatives of eq. (32) with respect to θ_i and θ_j^*.

Exercise 2.7. Use this technique to compute the integrals

$$\left(\prod_{k=1}^{n} \int \mathrm{d}\eta_k^* \int \mathrm{d}\eta_k \right) e^{-\underline{\eta}^\dagger A \, \underline{\eta}} \, \eta_i \eta_j^* \, , \tag{33}$$

$$\left(\prod_{k=1}^{n} \int \mathrm{d}\eta_k^* \int \mathrm{d}\eta_k \right) e^{-\underline{\eta}^\dagger A \, \underline{\eta}} \, \eta_i \eta_j^* \eta_l \eta_m^* \, . \tag{34}$$

Wick's theorem for Grassmann integrals has thus exactly the same form as in the bosonic case, but one needs to keep track of the minus signs which arise when variables are reordered.