1 Gaussian integrals and Wick's theorem

1.1 Gaussian integrals

Exercise 1.1. Compute the basic, one-dimensional Gaussian integral and show that it is given by

$$\int_{-\infty}^{\infty} \mathrm{d}x \, e^{-\frac{1}{2}\lambda x^2} = \sqrt{\frac{2\pi}{\lambda}} \, , \qquad \mathrm{Re}(\lambda) > 0 \, . \tag{1}$$

The standard trick is to take the square and use polar coordinates.

Exercise 1.2. Show that

$$\int_{-\infty}^{\infty} \mathrm{d}x \, e^{-\frac{1}{2}\lambda x^2 + ax} = \sqrt{\frac{2\pi}{\lambda}} \exp\left(\frac{a^2}{2\lambda}\right) , \qquad \mathrm{Re}(\lambda) > 0 \, . \tag{2}$$

Next, we turn to n-dimensional Gaussian integrals. We consider an element x of $\mathbb{R}^n$ and a symmetric $n \times n$ matrix M and define a quadratic form $x^T M x$. Integrating over $\mathbb{R}^n$, we obtain the Gaussian integral

$$\int \mathrm{d}^n x \, \exp\left(-\frac{1}{2} x^T M x\right) = \mathcal{N}^{-1} \, . \tag{3}$$

Exercise 1.3. Show that

$$\mathcal{N} = \sqrt{\frac{\det M}{(2\pi)^n}} \, . \tag{4}$$

What are the conditions on the matrix M for the integral in eq. (3) to exist?

Exercise 1.4. Similarly, show that for $a \in \mathbb{R}^n$ and if M^{-1} exists,

$$\mathcal{N} \int \mathrm{d}^n x \, \exp\left(-\frac{1}{2} x^T M x + a^T x\right) = \exp\left(\frac{1}{2} a^T M^{-1} a\right) \, . \tag{5}$$

Gaussian integrals appear in many areas of physics, for instance in statistical physics and in path-integral quantization. In Feynman's path-integral formulation of quantum mechanics, one needs to compute an integral of the form

$$\langle x_f, t_f \,|\, x_i, t_i \rangle = \int \mathcal{D}x(t) \, e^{iS[x(t)]} \tag{6}$$

to obtain the amplitude that a particle which was at point x_i at time t_i can be found at point x_f at time t_f, where $S[x(t)]$ is the action associated with the path $x(t)$ and the symbol

$\int\mathcal{D}x(t)$ indicates that one should sum over all paths $x(t)$ starting at $x(t_i) = x_i$ and ending at $x(t_f) = x_f$.

As it stands, it is not clear what eq. (6) means. To define it, one discretizes time $t_k = t_i + k\cdot\Delta$ such than $t_0 = t_i, t_{n+1} = t_f$. The integral over all paths takes the form

$$\int\mathcal{D}x(t) \longrightarrow \prod_{k=1}^{n}\int \mathrm{d}x_k = \int \mathrm{d}^n x\ , \tag{7}$$

where $x_k = x(t_k)$ is the position after k time steps. We see that we encounter integrals over $\mathbb{R}^n$ as in eq. (3). To evaluate the oscillatory expression (6), one first computes it for imaginary (Euclidean) time $t = -i\tau$ with $\tau \in \mathbb{R}$. Performing this so-called Wick rotation leads to $iS[x(t)] \to -S_E[x(\tau)]$ so that one can work with a real exponent as in eq. (3). After performing the integrals and taking $\Delta \to 0$ one then analytically continues the result back to physical time values.[2] Doing so, the path integral for a harmonic oscillator (or a free particle) boils down to the evaluation of Gaussian integrals like in eq. (3).

The path integral formulation plays an important role in Quantum Field Theory (QFT), where amongst others it forms the basis for numerical computations. In field theories one integrates over all field configurations $\phi(t,\vec{x})$ instead of the path $x(t)$. In discretized form, one then integrates over the field values $\phi_i \equiv \phi(t_i,\vec{x}_i)$. The discrete set of points $(t_i,\vec{x}_i)$ is called a lattice and to obtain the continuum result one takes the limit where the distance between lattice points (the lattice spacing) goes to zero.

1.2 Generating function, Wick's theorem

If we include the normalization factor $\mathcal{N}$, we can view the integrand in eq. (3), *viz.*

$$\rho(x) = \mathcal{N}\exp\left(-\frac{1}{2}x^T M x\right)\ , \tag{8}$$

as a probability distribution in $\mathbb{R}^n$ since it is normalized and strictly positive as long as M is a real, symmetric and positive[3] matrix. We can then compute expectation values as

$$\langle A(x)\rangle \equiv \int \mathrm{d}^n x\,\rho(x)\,A(x)\ . \tag{9}$$

The m-point correlation functions

$$\langle x_{i_1}\dots x_{i_m}\rangle \tag{10}$$

play an important role when one computes path integrals and we now analyze them in detail. The result is Wick's theorem, which provides the basis for perturbative computations in QFT.

To obtain the expectation values in eq. (10), let us consider ($b_i x_i \equiv \sum_i b_i x_i$)

$$\mathcal{Z}(b) \equiv \langle e^{b_i x_i}\rangle = \int \mathrm{d}^n x\,\rho(x) e^{b_i x_i} = \sum_{m\geq 0}\frac{1}{m!}\,b_{i_1}\dots b_{i_m}\int \mathrm{d}^n x\,\rho(x)\,x_{i_1}\dots x_{i_m}\ . \tag{11}$$

[2]It may be noted that the analytic continuation is not unique; different possibilities correspond to different time orderings of operator expectation values (see later sections).

[3]All its eigenvalues are strictly positive.

The quantity $\mathcal{Z}(b)$ is the *generating function* of moments of the probability distribution $\rho(x)$:

$$\mathcal{Z}(b) = \sum_{m\geq 0} \frac{1}{m!}\, b_{i_1} \ldots b_{i_m} \, \langle x_{i_1} \ldots x_{i_m} \rangle. \tag{12}$$

The inverse relation can be written as

$$\langle x_{i_1} \ldots x_{i_m} \rangle = \left[\frac{\partial}{\partial b_{i_1}} \cdots \frac{\partial}{\partial b_{i_m}} \mathcal{Z}(b) \right]_{b=0} . \tag{13}$$

In general, for an arbitrary probability density $\rho(x)$, $\mathcal{Z}(b)$ cannot be calculated exactly. But it can easily be evaluated for the Gaussian probability distribution. According to eq. (5), the generating function of the moments of the Gaussian distribution is

$$\begin{aligned} \mathcal{Z}(b) &= \mathcal{N} \int \mathrm{d}^n x \, \exp\left(-\frac{1}{2} x^T M x + b^T x \right) \\ &= \langle \exp(b^T x) \rangle \\ &= \exp\left(\frac{1}{2} b^T M^{-1} b \right) . \end{aligned} \tag{14}$$

The inverse relation is

$$\langle x_{i_1} \ldots x_{i_m} \rangle = \left[\frac{\partial}{\partial b_{i_1}} \cdots \frac{\partial}{\partial b_{i_m}} \exp\left\{ \frac{1}{2} b_i (M^{-1})_{ij} b_j \right\} \right]_{b=0} . \tag{15}$$

Exercise 1.5. Show that M is symmetric and, if M^{-1} exists, it is symmetric as well.

Exercise 1.6. Prove that:

$$\begin{aligned} \tfrac{\partial}{\partial b_{i_1}} \exp\left\{ \tfrac{1}{2} b_i (M^{-1})_{ij} b_j \right\} &= (M^{-1})_{i_1 k} b_k \exp\left\{ \tfrac{1}{2} b_i (M^{-1})_{ij} b_j \right\} , \\ \tfrac{\partial}{\partial b_{i_1}} \tfrac{\partial}{\partial b_{i_2}} \exp\left\{ \tfrac{1}{2} b_i (M^{-1})_{ij} b_j \right\} &= \Big[(M^{-1})_{i_1 i_2} + (M^{-1})_{i_1 k} b_k (M^{-1})_{i_2 l} b_l \Big] \exp\left\{ \tfrac{1}{2} b_i (M^{-1})_{ij} b_j \right\} . \end{aligned}$$

Exercise 1.7. Find a general expression for $\frac{\partial}{\partial b_{i_1}} \ldots \frac{\partial}{\partial b_{i_n}} \exp\left\{ \frac{1}{2} b_i (M^{-1})_{ij} b_j \right\}$. [Hint: Looking at the explicit expressions for the lowest few derivatives, you should observe a simple pattern, which can then be established using induction.]

Exercise 1.8. Use the previous results at $b = 0$, relevant for eq. (15), to verify that

$$\begin{aligned} \langle x_{i_1} x_{i_2} \rangle &= (M^{-1})_{i_1 i_2} , \\ \langle x_{i_1} x_{i_2} x_{i_3} x_{i_4} \rangle &= (M^{-1})_{i_1 i_2} (M^{-1})_{i_3 i_4} + (M^{-1})_{i_1 i_3} (M^{-1})_{i_2 i_4} + (M^{-1})_{i_1 i_4} (M^{-1})_{i_2 i_3} , \end{aligned} \tag{16}$$

and that the correlators of an odd number of points vanish,

$$\langle x_{i_1} \dots x_{i_{2m+1}} \rangle = 0\,. \tag{17}$$

Exercise 1.9. Derive *Wick's theorem*:

$$\langle x_{i_1} \dots x_{i_{2m}} \rangle = \sum_P \langle x_{k_1} x_{k_2} \rangle \dots \langle x_{k_{2m-1}} x_{k_{2m}} \rangle, \tag{18}$$

where the sum is over all pairings P, i.e. all possible ways to group the indices $i_1, i_2, \dots, i_{2m}$ into m pairs $(k_1, k_2), \dots, (k_{2m-1}, k_{2m})$. The theorem states that for the Gaussian integral all higher-point correlators reduce to products of the 2-point correlation function, which is given by the inverse of the matrix in the exponent of the Gaussian integral:

$$\langle x_{i_1} x_{i_2} \rangle = (M^{-1})_{i_1 i_2}\,. \tag{19}$$

(As usual in mathematics, the proof of the final theorem is trivial after you have prepared it with a highly non-trivial proof of a lemma, in our case the general form in Exercise 1.7.)