Exercises Electrodynamics	**Sheet 7**

[Tutorial 9.11.]

Exercise 1: Making use of spherical coordinates, we consider the vector potentials

$$\vec{A}^{(1)} := q_M \frac{1-\cos\theta}{r\sin\theta}\,\vec{e}_\varphi\,, \quad \vec{A}^{(2)} := -q_M \frac{1+\cos\theta}{r\sin\theta}\,\vec{e}_\varphi\,.$$

(a) Determine the magnetic inductions $\vec{B}^{(1,2)}$ that originate from these vector potentials.

(b) As the magnetic inductions are the same, the vector potentials should be related through a gauge transformation. Show that $\vec{A}^{(2)} = \vec{A}^{(1)} + \nabla\chi$, with $\chi = -2q_M\varphi$.

(c) The magnetic inductions found look as if they originated from a "monopole". This is confusing, since we thought that for magnetic fields the lowest term in the multipole expansion originates from a dipole. What could be wrong with the vector potentials that we have constructed?

Hint: in spherical coordinates,

$$\begin{aligned}
\nabla\chi &= \vec{e}_r\,\partial_r\chi + \frac{\vec{e}_\theta}{r}\,\partial_\theta\chi + \frac{\vec{e}_\varphi}{r\sin\theta}\,\partial_\varphi\chi\,, \\
\nabla\times\vec{A} &= \frac{\vec{e}_r}{r\sin\theta}\left[\partial_\theta(\sin\theta A_\varphi) - \partial_\varphi A_\theta\right] + \frac{\vec{e}_\theta}{r}\left[\frac{\partial_\varphi A_r}{\sin\theta} - \partial_r(rA_\varphi)\right] + \frac{\vec{e}_\varphi}{r}\left[\partial_r(rA_\theta) - \partial_\theta A_r\right]\,.
\end{aligned}$$

Exercise 2: If we consider quantum mechanics in the presence of electromagnetic fields, then the wave function gets transformed in a gauge transformation, as

$$\psi^{(2)}(\vec{x}) = e^{\frac{iq\chi(\vec{x})}{\hbar c}}\psi^{(1)}(\vec{x})\,, \quad \hbar \equiv \frac{h}{2\pi}\,.$$

Let us run a full circle around the origin, $\varphi_0 \to \varphi_0 + 2\pi$. The wave functions must be single-valued everywhere. Recalling the answer of exercise 1(b), show that this leads to the famous quantization condition of the electric charge that was discovered by Dirac,

$$q = n\,\frac{\hbar c}{2q_M}\,, \quad n \in \mathbb{Z}\,.$$

Exercise 3: Solve the wave equation

$$\Box f(\vec{x}, t) = 0$$

by making a Fourier transformation in all coordinates (both space and time).