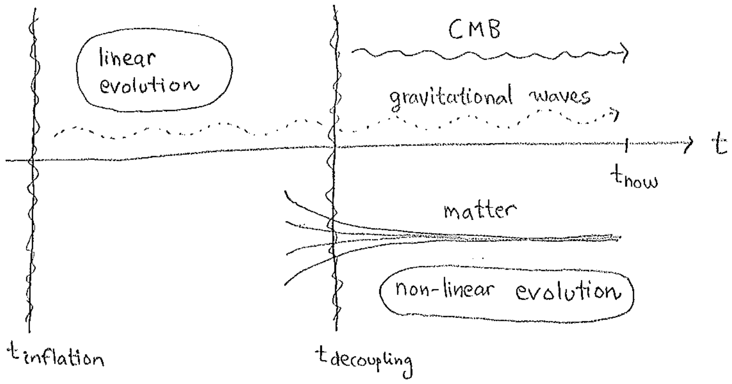
Temperature anisotropies, structure formation

Overall picture: The fluctuations that are generated for ϕ during inflation turn later on into physical perturbations in temperature and density.* They also „evolve" during this process. The most important epochs:

* e.g. V.F. Mukhanov, H.A. Feldman, R.H. Brandenberger, „Theory of cosmological perturbations", Phys. Rep. 215 (1992) 203



In order to describe the linear part of the evolution, we linearize Einstein equations around the homogeneous Friedmann case (p.1), by writing**

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + g'_{\mu\nu}$$

parametrized by a scalar, vector and tensor part

$$T_{\mu\nu} = \bar{T}_{\mu\nu} + T'_{\mu\nu}$$

parametrized by perturbations of e, p

** More precisely: before reheating $T'_{\mu\nu}$ is parametrized by ϕ', after reheating by a thermal e', p'. So:

$$T'_{\mu\nu}[\phi'] \xrightarrow{\text{during inflation}} g'_{\mu\nu} \xrightarrow{\text{after reheating}} T'_{\mu\nu}[e', p']$$

The initial ϕ' sources a perturbation of the scalar part of $g'_{\mu\nu}$, resulting in a „curvature perturbation"; this in turn sources a density perturbation. The ϕ'-perturbation itself is erased during reheating (p.46), but has already left its mark. The relation of the final density perturbation spectrum $P_\delta(\vec{k})$ to the original ϕ-spectrum $P_\phi(\vec{k})$ from p.48 is called a „transfer function".

Let us summarize the role played by different parts of $g'_{\mu\nu}$ as follows:

(i) the scalar part couples to density perturbations and is thus ultimately responsible for structure formation

(ii) the vector part decays in time, and is not sourced by simple inflationary models; it can thus most probably be neglected.

(iii) the tensor part corresponds to gravitational waves; the production of a primordial gravitational spectrum is yet another important prediction of inflation.

The physical observables are characterized by power spectra like for ϕ on p.47, i.e.

$$\langle [T'^2(\vec{x})] \rangle = \int d\ln k \; P_T(\vec{k}) \,,$$

$$\langle [e'^2(\vec{x})] \rangle = \int d\ln k \; P_\delta(\vec{k}) \,,$$

$$\langle [h_{TT}^2(\vec{x})] \rangle = \int d\ln k \; P_h(\vec{k}) \,.$$

Jeans instability: In order to understand when fluctuations start to grow, let us consider small perturbations of a medium, i.e. sound waves. For this consideration we can work with local Minkowskian coordinates.

P.40 (without ϕ which has decayed): $T^{\mu\nu} = (e+p)u^\mu u^\nu - p g^{\mu\nu}$.

Consider „longitudinal" perturbations of a wave moving in z-direction. We write $e = \bar{e} + e'$, $p = \bar{p} + p'$, $u = (1, v'\vec{e}_z)$, and expand to linear order:

$$\partial_0 T^{00} + \partial_z T^{z0} = \partial_t e' + \partial_z [(\bar{e}+\bar{p})v'] = 0 \qquad (i)$$

$$\partial_0 T^{0z} + \partial_z T^{zz} = \partial_t [(\bar{e}+\bar{p})v] + \partial_z [p' + O(v^2)] = 0 \,. \qquad (ii)$$

Now, what if this takes place in a gravitational potential Φ? Should do a GR analysis, but essentially the „force" $-\nabla p$ is supplemented by the Newton force $-\bar{e}\nabla\Phi$, so add this on the right-hand side of (ii). Then we consider

$$0 = \partial_t (i) = \partial_t^2 e' + \partial_t \partial_z [(\bar{e}+\bar{p})v']$$

$$\overset{\text{insert } \partial_z(ii)}{=} \partial_t^2 e' - \partial_z^2 p' - \bar{e}\,\partial_z^2 \Phi \,.$$

Now we write $\partial_z^2 p' = \partial_z \left(\frac{\partial \bar{p}}{\partial \bar{e}} \frac{\partial e'}{\partial z} \right) = \underbrace{\frac{d\bar{p}/dT}{d\bar{e}/dT}}_{\text{p.2: } c_s^2} \partial_z^2 e'$.

In addition, Φ satisfies $\nabla^2 \Phi = 4\pi G_N e' = \frac{4\pi}{m_{Pl}^2} e'$.
Finally, we go to momentum space, $e'(t,z) \to e'(t)\, e^{ik_z z}$.

$$\Rightarrow \quad \partial_t^2 e'(t) + \left(c_s^2 k_z^2 - \frac{4\pi\bar{e}}{m_{Pl}^2} \right) e'(t) = 0 \,.$$

If the parentheses become negative, there is an exponentially growing mode, i.e. the system is unstable to gravitational collapse:

$$c_s k_z < \frac{\sqrt{4\pi\bar{e}}}{m_{Pl}} \quad \overset{k_z = \frac{2\pi}{\lambda}}{\Longrightarrow} \quad \lambda > \frac{2\pi c_s m_{Pl}}{\sqrt{4\pi\bar{e}}} \,.$$

But we recall that $H = \sqrt{\frac{8\pi}{3}\frac{\bar{e}}{m_{Pl}^2}}$ (F1), so there is instability if

$$\boxed{\lambda \gtrsim \frac{2\pi c_s}{H} \,.}$$

Consider now two regimes:

(a) „relativistic matter" $\overset{\text{p.2}}{\Longrightarrow}$ $c_s \approx \frac{1}{\sqrt{3}}$ $\Rightarrow$ only superhorizon modes could grow, but for them our Minkowskian analysis is not reliable $\Rightarrow$ no exponential growth.

(b) „non-relativistic matter" $\Rightarrow$ $c_s \ll 1$ $\Rightarrow$ instability can start on subhorizon scales.

For „normal" Standard Model matter, instabilities can therefore start to grow only once photons have decoupled! For dark matter, which does not couple to medium, earlier growth is possible.

Temperature anisotropies: Let us now focus on the anisotropies in CMB. Roughly speaking, these anisotropies are useful because they are small ($\delta T/T \sim 10^{-5}$)*, and only sensitive to linear evolution. Therefore they have a relatively simple relation to initial perturbations (even if in a quantitative analysis the propagation of photons in an anisotropic gravitational field, the so-called Sachs-Wolfe effect, plays a role).

* COBE collaboration, "Structure in the COBE differential microwave radiometer first year maps", Astrophys. J. 396 (1992) L1

On a qualitative level, we can understand the transfer function from P_ϕ to P_T as follows. From p. 2: $s a^3 = \text{const.}$, $s \propto g_* T^3$

$$\Rightarrow \quad \frac{\delta T}{T} \sim -\frac{\delta a}{a}.$$

Now we write $\delta a \sim \frac{da}{dt}\frac{dt}{d\phi}\,\delta\phi$ and recall from p.45 that in the slow-roll regime $3H\dot\phi \approx -V'$ and $H^2 \approx \frac{8\pi}{3}\frac{V}{m_{Pl}^2}$. Therefore,

$$\frac{\delta T}{T} \sim -\frac{H}{\dot\phi}\,\delta\phi\,,$$

$$\Rightarrow \quad \frac{P_T}{T^2} \sim \frac{H^2}{\dot\phi^2} P_\phi \approx \frac{9H^6}{4\pi^2 (3H\dot\phi)^2} \approx \underbrace{\frac{9}{4\pi^2}\left(\frac{8\pi}{3}\right)^3}_{\frac{128\pi}{3}} \frac{V^3}{m_{Pl}^6 (V')^2} \sim (10^{-5})^2$$

p.48: $P_\phi \approx \frac{H^2}{4\pi^2}$ — COBE result

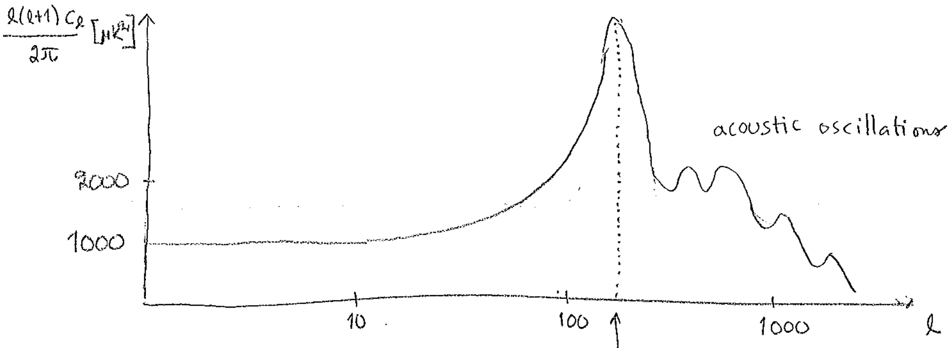
For instance, the model discussed on p. 46 has $\frac{V^3}{(V')^2} = \frac{m^2\phi^4}{8}$, and we know that $\phi(0) \gtrsim 5\, m_{Pl}$, so we now get $\frac{m}{m_{Pl}} \sim 10^{-7}$, i.e. a strong constraint on the inflationary potential.

The observed data on temperature anisotropies is parametrized as

$$\delta T = \sum_{\ell,m} a_{\ell m} Y_{\ell m}(\theta,\varphi)\,,$$

and one conventionally defines $C_\ell := \frac{1}{2\ell+1}\sum_{m=-\ell}^{\ell} |a_{\ell m}|^2$.

The basic plot:



The maximum corresponds to horizon radius at decoupling. It is at $\ell \sim 200$, $\theta \sim 2°$. As the decoupling happened at $t \sim 300\,000\,y$ and as $pc \sim 3\,ly$, the radius was $\sim 10^5\,pc$ at $T \approx 3000\,K$ (p.10), and is $\frac{a(t_{now})}{a(t_{decoupling})} \approx \frac{T_{decoupling}}{T_{now}} \approx \frac{3000K}{3K} = 10^3$ larger today, i.e. 100 Mpc.

(cf. e.g. p. 20)

Fitting the above shape to numerical simulations leads to the currently most precise determinations** of the energy densities carried by baryons, dark matter and cosmological constant, as well as to constraints on the $|\vec{k}|$-(in)dependence of $P_\phi(\vec{k})$.

** Planck collaboration, "Planck 2018 results. VI. Cosmological parameters", arXiv: 1807.06209

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Large-scale structure: We finally return to density perturbations that start to grow rapidly after recombination. It turns out* that an important moment here is the transition from „radiation-dominated" to „matter-dominated" expansion:

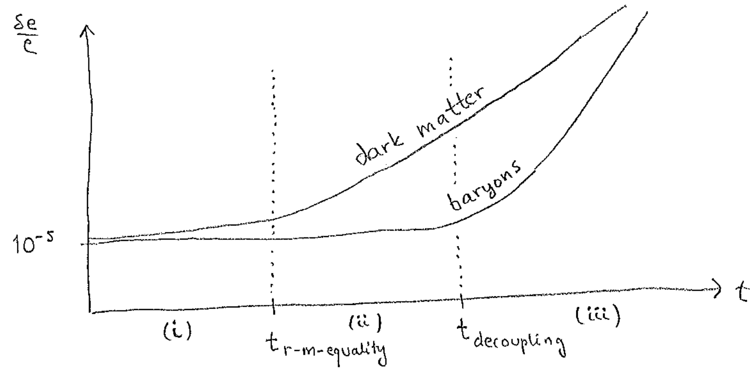
$$e_r \approx \frac{\pi^2}{30}\left[2 + \frac{7}{4}\left(\frac{4}{11}\right)^{4/3} N_{\rm eff}\right] T^4 \;, \quad e_m = e_{DM} + m_p \cdot \frac{n_B}{n_\gamma} \cdot n_\gamma \;.$$

photons and neutrinos, p. 7

cf. p. 9

* The reason is partly just that c_s^2 decreases, as discussed on p. 50, but this is not the only effect.

If $e_{DM} = 0$, this yields $T \sim 0.1\,\text{eV}$; with e_{DM}, $T \sim \text{eV}$. Sketch:

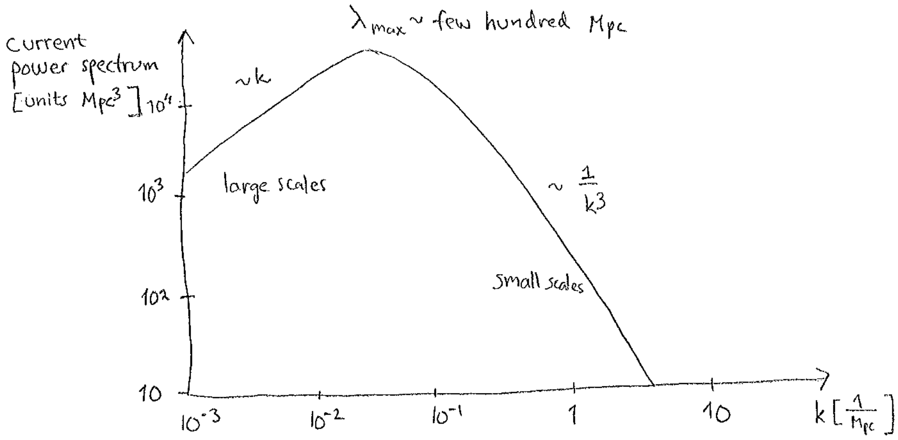


Summary of the different epochs:

(i) radiation dominated era: expansion too rapid (i.e. „friction" from H too large) ⇒ prevents growth, apart from logarithmic effects for dark matter

(ii) matter dominated era before recombination: perturbations in dark matter start to grow. Baryons and photons oscillate in dark matter potential wells: „acoustic oscillations".

(iii) matter dominated era after recombination: photons decouple, stream freely, and constitute CMB. Baryonic perturbations catch up with dark matter and grow to form the present structure.



Study of the non-linear regime is challenging, requiring large-scale simulations, but could be important for instance for determining neutrino masses (cf. p. 8).

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