Inflation and origin of density perturbations

Problems addressed: There are a number of „unnatural" observations that call for an explanation:

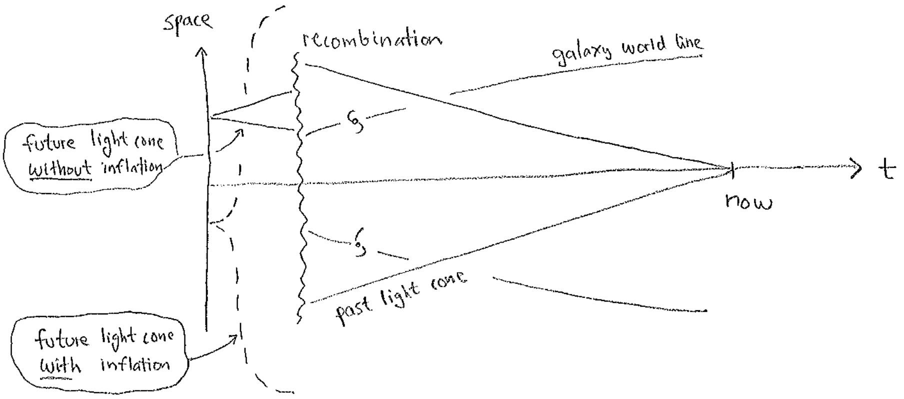
„qualitative"

- * „flatness problem": empirically, $\frac{k}{a^2} \ll \frac{\dot{a}^2}{a^2}$ in eq. (F1) on p. 1, so the value of k is difficult to determine. Moreover, extrapolating current data backwards in time, the hierarchy was even bigger at earlier times, $\frac{k}{a^2} \lll \frac{\dot{a}^2}{a^2}$ as t decreases.
- * „horizon problem": the last scattering surface consists of $\sim 10^5$ regions that should have been causally disconnected back then. How come their temperatures only differ by $\delta T/T \sim 10^{-5}$?

„quantitative"

- * where did the density perturbations that later collapse into galaxy clusters and galaxies come from?

Basic idea: The problems above can be solved by an early period of exponential expansion*:



*A.H. Guth, „Inflationary universe: a possible solution to the horizon and flatness problems", Phys. Rev. D 23 (1981) 347.

History is more complicated on the third issue (perturbations).

Implementation: Consider a spatially homogeneous scalar field ϕ. According to p. 40, it carries the energy density $e = T^0{}_0 = \frac{\dot{\phi}^2}{2} + V(\phi)$ and pressure $p = -T^1{}_1 = \frac{\dot{\phi}^2}{2} - V(\phi)$. It follows that $\dot{e} = \dot{\phi}(\ddot{\phi} + V'(\phi))$. Inserting into (F1) & (F3) from p. 1, the Friedmann equations take the form

$$\begin{cases} \dfrac{\dot{a}^2}{a^2} + \dfrac{k}{a^2} = \dfrac{8\pi}{3 m_{Pl}^2}\left[\dfrac{\dot{\phi}^2}{2} + V(\phi)\right] & \text{(F1)} \\ \ddot{\phi} + 3H\dot{\phi} = -V'(\phi) & \text{(F3)} \end{cases}$$

This is a complicated non-linear system of differential equations, depending also on initial conditions for $a, \phi, \dot{\phi}$. But let us assume that we find ourselves** in a „slow-roll regime":

** Why so? Perhaps there is an anthropic explanation? In any case, the slow-roll regime is said to be an „attractor solution", i.e. independent of the precise initial values, and losing memory of them fast.

(i) $\dot{\phi}^2 \ll V(\phi)$ [kinetic energy small compared with potential energy]

(ii) $|\ddot{\phi}| \ll |3H\dot{\phi}| \approx |V'(\phi)|$ [large „friction", i.e. overdamped regime]

Also, we make the ansatz $\frac{k}{a^2} \ll \frac{\dot{a}^2}{a^2}$, to be verified a posteriori. Then

$$\frac{\dot{a}}{a} = \sqrt{\frac{8\pi}{3}}\,\frac{\sqrt{V}}{m_{Pl}} \approx \text{const} =: H \quad \Rightarrow \quad a(t) = a(0)\, e^{Ht} \ \%$$

$$\Rightarrow \quad \frac{k}{a^2} = \frac{k}{a^2(0)}\, e^{-2Ht} = \text{exponentially small } \%$$

How should $V(\phi)$ look like in order for the assumptions to be consistent?

$$\text{(i)} \Rightarrow \frac{\dot{\phi}^2}{V} \underset{\text{(F3) with (ii)}}{\approx} \frac{(V')^2}{9H^2 V} \underset{\text{(F1) with (i)}}{\approx} \frac{m_{Pl}^2}{24\pi}\left(\frac{V'}{V}\right)^2 \ll 1 \qquad \text{(i')}$$

$$* \quad \dot{\phi}^2 \approx \frac{m_{Pl}^2}{24\pi}\frac{(V')^2}{V} \Rightarrow 2\dot{\phi}\ddot{\phi} \approx \frac{m_{Pl}^2 \dot{\phi} V'}{24\pi}\left\{\frac{2V''}{V} - \left(\frac{V'}{V}\right)^2\right\} \Rightarrow \frac{\ddot{\phi}}{V'} \approx \underbrace{\frac{m_{Pl}^2}{24\pi}\left\{\frac{V''}{V} - \frac{1}{2}\left(\frac{V'}{V}\right)^2\right\}}_{\ll 1 \ \text{(ii')}}$$

Example:

*A.D. Linde, „Chaotic inflation", Phys. Lett. B 129 (1983) 177

Consider the simplest possible potential, corresponding to a free massive scalar field*:

$$V(\phi) := \frac{1}{2} m^2 \phi^2 .$$

Despite the extreme simplicity of the model, the full equations (F1) & (F3) are complicated enough that they can only be solved numerically. But let us assume that we are in the slow-roll regime. What does this require?

(i') from p.45 : $\frac{m_{Pl}^2}{24\pi}\left(\frac{2\phi}{\phi^2}\right)^2 \ll 1 \Rightarrow \phi \gg \frac{m_{Pl}}{\sqrt{6\pi}}$

(ii') from p.45 : $\frac{m_{Pl}^2}{24\pi}\,\frac{2}{\phi^2} \ll 1 \Rightarrow$ essentially the same criterion

The requirement of having $\phi \gtrsim m_{Pl}$ may sound dangerous, however if $\phi \ll \frac{m_{Pl}^2}{m}$, the energy density is $\ll m_{Pl}^4$, and we may hope that quantum gravity is not yet important.

We may now solve (F1) & (F3) in the slow-roll regime:

$$(F1) \Rightarrow H \approx \sqrt{\frac{4\pi}{3}}\,\frac{m\phi}{m_{Pl}}$$

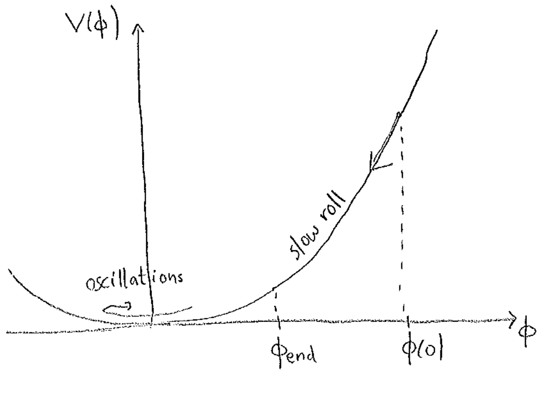
$$(F3) \Rightarrow 3H\dot{\phi} \approx -m^2\phi$$

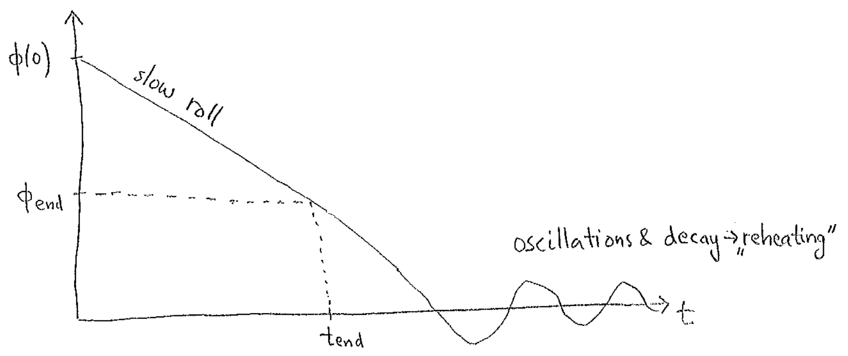
Dividing the two we get $\sqrt{12\pi}\,\dot{\phi} \approx -m\,m_{Pl}$

$$\Rightarrow \phi(t) = \phi(0) - \frac{m\,m_{Pl}}{\sqrt{12\pi}}\,t , \quad 0<t<t_{end} .$$

non-oscillatory „slow" evolution.

The approximation breaks down when $\phi(t_{end}) \sim \frac{m_{Pl}}{\sqrt{6\pi}}$. Sketch:





At the very end ϕ loses its energy to radiation, and the universe „reheats" to a temperature T_r, given by

$$V(\phi_{end}) \sim g_* \frac{\pi^2}{30} T_r^4 .$$

An important characteristic of the slow-roll regime is the „number of e-folds" of the exponential expansion:

$$\frac{d\ln a}{d\phi} = \frac{1}{a}\frac{da}{d\phi} = \frac{1}{a}\frac{\dot{a}}{\dot{\phi}} = \frac{H}{\dot{\phi}} \underset{(3H\dot{\phi}\approx -V')}{\approx} \frac{-3H^2}{V'} \approx -\frac{8\pi}{m_{Pl}^2}\frac{V}{V'}$$

$$\Rightarrow N \approx -\frac{8\pi}{m_{Pl}^2}\int_{\phi(0)}^{\phi_{end}} \frac{V}{V'}\,d\phi = -\frac{4\pi}{m_{Pl}^2}\int_{\phi(0)}^{\phi_{end}} \phi\,d\phi = \frac{2\pi}{m_{Pl}^2}\left[\phi(0)^2 - \phi_{end}^2\right] \underset{\text{empirically needed}}{\gg} 50 .$$

E.g. $\phi(0) \approx 5\,m_{Pl}$ would be sufficient.

Fluctuations: We now move on to consider the ultimately most important aspect of inflation, namely how classical density perturbations get generated out of quantum fluctuations of ϕ. This is a subtle topic (quantum field theory in a curved background) and we only try to understand the basic point.

To get started, let us consider ϕ in flat spacetime ($H=0$). Like on p. 31, write $\phi = \bar\phi + \phi'$, and let $\bar\phi$ be a constant. If we are at the minimum of $V(\bar\phi)$, i.e. $V'(\bar\phi)=0$, and denote $m^2 = V''(\bar\phi)$, ϕ' satisfies the Klein-Gordon equation

$$\ddot\phi' - \nabla^2\phi' + m^2\phi' = 0 \, .$$

We are interested in computing the „spectrum" (i.e. Fourier transform) of equal-time fluctuations:

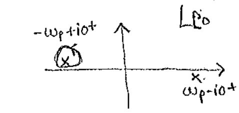
$$\Delta(\vec k) := \int d^3\vec x \, e^{-i\vec k\cdot(\vec x-\vec y)} \langle 0|\phi'(\vec x)\phi'(\vec y)|0\rangle \, .$$

In flat space-time two different approaches are possible:

(a) Feynman propagator [equal-time = time-ordered!]:

$$\int_{\vec x} e^{-i\vec k\cdot(\vec x-\vec y)} \int_{\vec p}\int_{-\infty}^{\infty}\frac{dp_0}{2\pi}\,\frac{i}{p_0^2-\vec p^{\,2}-m^2+i0^+}\, e^{i\vec p\cdot(\vec x-\vec y)}$$

$$= \int_{-\infty}^{\infty}\frac{dp_0}{2\pi}\cdot\int\frac{d^3\vec p}{(2\pi)^3}\,\frac{i}{p_0^2-\omega_p^2+i0^+}\,(2\pi)^3\delta^{(3)}(\vec p-\vec k)$$



$$= \frac{i}{2\pi}\cdot 2\pi i\cdot\frac{1}{(-2\omega_k)} = \frac{1}{2\omega_k} \, , \quad \omega_k := \sqrt{\vec k^2+m^2}$$

(b) Canonical quantization

Like on p. 22*: $\hat\phi' = \int\frac{d^3\vec p}{\sqrt{(2\pi)^3 2\omega_p}}\left(\hat a_p e^{-ip\cdot x} + \hat a_p^{+} e^{ip\cdot x}\right)$

$$\Rightarrow \langle 0|\hat\phi'(\vec x)\hat\phi'(\vec y)|0\rangle = \int\frac{d^3\vec p\, d^3\vec q}{(2\pi)^3 2\sqrt{\omega_p\omega_q}}\langle 0|\left(\hat a_p e^{i\vec p\cdot\vec x} + \cancel{\hat a_p^{+} e^{-i\vec p\cdot\vec x}}\right)\left(\cancel{\hat a_q e^{i\vec q\cdot\vec y}} + \hat a_q^{+} e^{-i\vec q\cdot\vec y}\right)|0\rangle$$

$$= \int\frac{d^3\vec p\, d^3\vec q}{(2\pi)^3 2\sqrt{\omega_p\omega_q}}\underbrace{\langle 0|[\hat a_p,\hat a_q^{+}]|0\rangle}_{\delta^{(3)}(\vec p-\vec q)} e^{i\vec p\cdot\vec x - i\vec q\cdot\vec y}$$

$$= \int\frac{d^3\vec p}{(2\pi)^3 2\omega_p}\, e^{i\vec p\cdot(\vec x-\vec y)}$$

$$\Rightarrow \Delta(\vec k) = \frac{1}{2\omega_k} \quad \% \, .$$

*But now with a real scalar field.

If we wanted to consider root-mean-squared fluctuations at the position $\vec x$, these could be obtained from the inverse transform:

$$\langle 0|[\phi'(\vec x)]^2|0\rangle = \lim_{\vec x\to\vec y}\int\frac{d^3\vec k}{(2\pi)^3}\, e^{i\vec k\cdot(\vec x-\vec y)}\Delta(\vec k) = \int\frac{d^3\vec k}{(2\pi)^3}\cdot\frac{1}{2\omega_k} \, .$$

Recalling that $d^3\vec k = 4\pi k^2 dk = 4\pi k^3\, d\ln k$, the power spectrum of ϕ is conventionally defined as

$$P_\phi(\vec k) := \frac{|\vec k|^3}{2\pi^2}\cdot\Delta(\vec k) = \frac{|\vec k|^3}{2\pi^2}\cdot\frac{1}{2\omega_k} \, . \qquad (**)$$

for flat spacetime

Expanding background: We now generalize the consideration of p. 47 to inflation. After linearizing around (F3), the perturbations satisfy

$$\ddot{\phi}' + 3H\dot{\phi}' - \frac{1}{a^2}\nabla^2\phi' + V''(\bar{\phi})\,\phi' = 0 \,.$$

Slowly varying ⇒ treat as constant

very fast time dependence

slowly varying ⇒ treat as constant $=: m^2$

The explicit time dependence makes the definition of a vacuum state $|0\rangle$ ambiguous. Nevertheless we may take an ansatz like on p. 47, just replacing $\frac{1}{\sqrt{2\omega_p}}e^{-i\omega_p t} \to u(t)$:

$$\hat{\phi}' = \int \frac{d^3\vec{p}}{\sqrt{(2\pi)^3}} \left[u(t,p)\,\hat{a}_p\, e^{i\vec{p}\cdot\vec{x}} + u^*(t,p)\,\hat{a}_p^\dagger\, e^{-i\vec{p}\cdot\vec{x}} \right].$$

Then eq. (**) from p. 47 suggests

$$P_\phi(\vec{k}) = \frac{|\vec{k}|^3}{2\pi^2}\,|u(t,k)|^2 \,,$$

$$\ddot{u} + 3H\dot{u} + \left(\frac{k^2}{a^2} + m^2\right)u = 0 \,.$$

Let us now take a momentum mode with $|\vec{k}| \gg H$ initially. As time goes by, $\frac{|\vec{k}|}{a}$ decreases, and in the end $\frac{|\vec{k}|}{a} < H$. We say that this mode „exits the horizon". Later on, when inflation has ended, it „re-enters the horizon".

When the mode is beyond horizon, we may omit $\frac{|\vec{k}|^2}{a^2}$. In addition, the slow-roll conditions from p. 45 imply that

$$m^2 = V'' \overset{(ii')}{\ll} \frac{24\pi V}{m_{Pl}^2} \overset{(F1)}{\approx} 9H^2 \,, \quad \text{i.e.} \quad m \ll H \,.$$

So in this regime

$$\ddot{u} + 3H\dot{u} \approx 0 \quad \Rightarrow \quad u = c_1 + \underbrace{c_2 e^{-3Ht}}_{\text{decays fast!}}$$

So u becomes constant. This is nice, because then $P_\phi(\vec{k})$ is time-independent. However, to determine the value of c_1 requires a more precise solution, with the requirement that at early times (when $\frac{|\vec{k}|}{a} \gg H$) we match the flat-space solution $\frac{1}{\sqrt{2\omega_p}}e^{-i\omega_p t}$.

Result: $P_\phi(\vec{k}) \approx \left(\frac{H}{2\pi}\right)^2 \Big|_{|\vec{k}| = aH \text{ (at horizon exit)}}$ (*)

It is said that vacuum fluctuations „freeze" (while t-independent) and become „classical" (H is a macroscopic quantity).**

Because (*) does not depend on $|\vec{k}|$ if H is constant, it is called a „flat" spectrum, also known as a Harrison-Zeldovich spectrum.

** A physical interpretation: the vacuum energy density causing the exponential expansion is converted into large fluctuations.