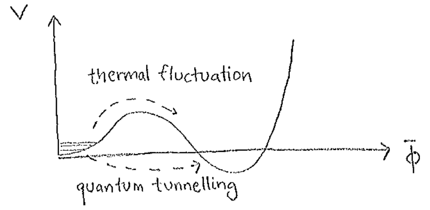
Thermal phase transitions. 3. Nucleation and dynamics

Semiclassical picture:* Assume now that the system has a first order phase transition, and consider how it proceeds in real time. Idea:

* S.R. Coleman, „The fate of the false vacuum. 1. Semiclassical theory", Phys. Rev. D 15 (1977) 2929



Let us use boundary conditions at spatial infinity, $\lim_{|\vec{x}|\to\infty} \phi(\vec{x}) = 0$, to define metastable energy „eigen states", and inspect their time evolution:

$$|\phi(t)\rangle = e^{-iEt}|\phi(0)\rangle = e^{-i[\mathrm{Re}E + i\,\mathrm{Im}E]t}|\phi(0)\rangle$$

$$\Rightarrow \quad \langle\phi(t)|\phi(t)\rangle = e^{2\,\mathrm{Im}E\,t}\langle\phi(0)|\phi(0)\rangle$$

$$\Rightarrow \quad \Gamma(E) = -2\,\mathrm{Im}E\,.$$

In a thermal ensemble, we might similarly expect $\langle\Gamma\rangle \simeq -2\,\mathrm{Im}F$. Look at the imaginary-time path integral (p.31):

$$F = -T\ln \int_{\text{b.c.}} \mathcal{D}\phi\, e^{-S_E[\phi]}\,.$$

Assume that there are two saddle points: $\phi = 0$ and $\phi = \hat{\phi}(\tau,\vec{x})$:

$$\frac{\delta S_E}{\delta\phi}\bigg|_{\phi=\hat{\phi}} = 0\,, \quad \hat{\phi}(0,\vec{x}) = \hat{\phi}(\beta,\vec{x})\,, \quad \lim_{|\vec{x}|\to\infty}\hat{\phi}(\tau,\vec{x}) = 0\,.$$

The fluctuation operator around $\hat{\phi}$ could have a negative eigenmode:**

** This corresponds to an unstable direction around a saddle point.

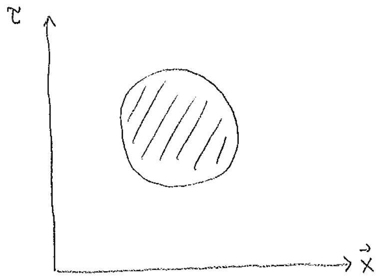
$$\frac{\delta^2 S_E}{\delta\phi^2}\bigg|_{\phi=\hat{\phi}} f_-(\tau,\vec{x}) = -\lambda_-^2 f_-(\tau,\vec{x})$$

$$\Rightarrow \quad F \simeq -T\ln\Big\{\underbrace{Z[\phi=0]}_{\text{„large"}} + \underbrace{e^{-S_E[\hat{\phi}]}}_{\text{„small"}}\underbrace{\int df_-\, e^{\frac{1}{2}\lambda_-^2 f_-^2}}_{\simeq\, i\sqrt{\frac{2\pi}{\lambda_-^2}}}\int\prod_{n\geq 0} df_n\, e^{-\frac{1}{2}\lambda_n^2 f_n^2}\Big\}$$

$$\Rightarrow \quad \Gamma \simeq \frac{T}{Z[\phi=0]}\cdot\frac{e^{-S_E[\hat{\phi}]}}{\left|\det\left(\frac{\delta^2 S_E[\hat{\phi}]}{\delta\phi^2}\right)\right|^{1/2}}\,.$$

It turns out that the precise definition of the „prefactor" is subtle. Here we focus on $S_E[\hat{\phi}]$, often sufficient for practical purposes. The „shape" of the solution can be sketched as follows:***

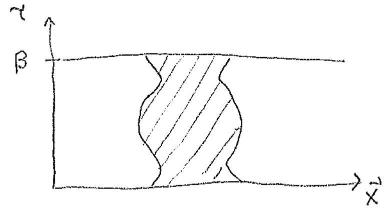
*** A.D. Linde, „Fate of the false vacuum at finite temperature: theory and applications", Phys. Lett. B 100 (1981) 37



$T = 0$

$\Rightarrow$ 4d symmetry

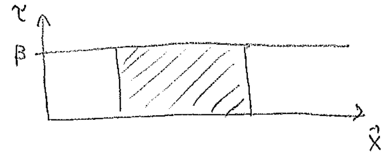
$\Rightarrow$ „instanton" for quantum tunnelling



$T \neq 0$

$\Rightarrow$ (3+1)d symmetry

$\Rightarrow$ „caloron" for thermally modified quantum tunnelling



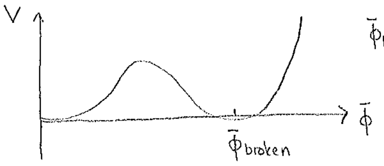
$T \gg 0$

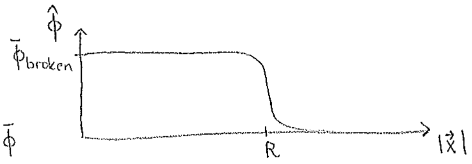
$\Rightarrow$ 3d symmetry

$\Rightarrow$ $S_E[\hat{\phi}] = \beta\int d^3\vec{x}\, L_E^{(n=0)}$

$\Rightarrow$ classical thermal fluctuation

<u>Classical tunnelling action:</u> Consider the situation just below T_c:





Imaginary-time Lagrangian: $L_E^{(n=0)} = \frac{1}{2}(\partial_i \hat{\phi})^2 + V(\hat{\phi})$.

Saddle-point equation of motion (assuming spherical symmetry; $r := |\vec{x}|$):

$$\frac{d^2\hat{\phi}}{dr^2} + \frac{2}{r}\frac{d\hat{\phi}}{dr} = V'(\hat{\phi}) \quad ; \quad \hat{\phi}'(0) = 0 \quad ; \quad \hat{\phi}(\infty) = 0 .$$

Let us first inspect the region $r \simeq R$, where $\frac{2\hat{\phi}'}{R}$ is small:

$$\frac{d^2\hat{\phi}}{dr^2} \simeq V'(\hat{\phi}) \Rightarrow \frac{1}{2}(\hat{\phi}')^2 \simeq V(\hat{\phi}) - V(\bar{\phi}_{\text{broken}}) .$$

multiply both sides by $\hat{\phi}'$ and integrate

We can now write the saddle point (i.e. nucleation) action as

$$S_E[\hat{\phi}] = \beta \int d^3\vec{x} \left\{ \frac{1}{2}\hat{\phi}'^2 + V(\hat{\phi}) \right\}$$

$$\simeq \beta 4\pi \left\{ \int_0^{R-\delta} dr\, r^2 \left[V(\bar{\phi}_{\text{broken}}) \right] + \int_{R-\delta}^{R+\delta} dr\, r^2 \left[\frac{1}{2}\hat{\phi}'^2 + V(\hat{\phi}) \right] + \int_{R+\delta}^{\infty} dr\, r^2 \left[0 \right] \right\}$$

$$\simeq \beta 4\pi \left\{ \int_0^{R+\delta} dr\, r^2 \left[V(\bar{\phi}_{\text{broken}}) \right] + \underbrace{\int_{R-\delta}^{R+\delta} dr\, r^2 \left[\frac{1}{2}\hat{\phi}'^2 + V(\hat{\phi}) - V(\bar{\phi}_{\text{broken}}) \right]} \right\}$$

$$\simeq R^2 \int_{R-\delta}^{R+\delta} dr \left(\frac{d\hat{\phi}}{dr} \right)^2 = R^2 \int_0^{\bar{\phi}_{\text{broken}}} d\hat{\phi}\, \frac{d\hat{\phi}}{dr}$$

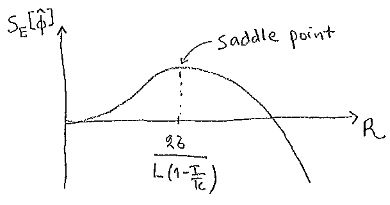
$$\simeq R^2 \int_0^{\bar{\phi}_{\text{broken}}} d\hat{\phi}\, \sqrt{2\left[V(\hat{\phi}) - V(\bar{\phi}_{\text{broken}}) \right]}$$

We call $\sigma := \int_0^{\bar{\phi}_{\text{broken}}} d\hat{\phi} \sqrt{2[\ldots]}$ the <u>surface tension</u>. Moreover the free energy difference can be expressed close to T_c as

$$V_{\text{eff}}(\bar{\phi}_{\text{broken}}) = -L\left(1 - \frac{T}{T_c}\right) + O(T - T_c)^2 ,$$

where L is called the <u>latent heat</u>. Then the action is

$$S_E[\hat{\phi}] \simeq \beta \left\{ -\frac{4\pi}{3} R^3 L \left(1 - \frac{T}{T_c}\right) + 4\pi R^2 \sigma \right\} .$$



Now we extremize with respect to $R \Rightarrow R = \frac{8\pi\sigma}{4\pi L(1-\frac{T}{T_c})} = \frac{2\sigma}{L(1-\frac{T}{T_c})}$.

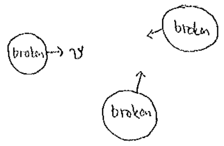
Inserting this back to the action yields

$$S_E[\hat{\phi}] \simeq \beta \cdot 4\pi \cdot \left(-\frac{8}{3} + 4 \right) \frac{\sigma^3}{L^2 (1 - \frac{T}{T_c})^2} = \frac{16\pi}{3} \frac{\sigma^3 \beta}{L^2 (1 - \frac{T}{T_c})^2} .$$

In summary, $S_E[\hat{\phi}] \to \infty$ for $T \to T_c$, so that $e^{-S_E[\hat{\phi}]} \to 0$.

This implies that <u>nucleation can only happen after supercooling</u>.

Bubble dynamics: The overall picture of what happens after nucleation is as follows:



bubbles nucleate at $T \approx T_n < T_c$ with radii $\gtrsim R_c(T_n)$ and start then to grow with velocity v

Let us denote the nucleation probability per time and volume by $p = p_0 e^{-\hat{S}_E}$.

We expand $\hat{S}_E(t)$ around an effective nucleation time t_n:

$$\hat{S}_E(t) \approx \hat{S}_E(t_n) + \hat{S}_E'(t_n)(t-t_n) \; ; \quad \hat{S}_E'(t_n) < 0 \quad (\text{cf. p. 38}).$$

Let v be an effective velocity at which further nucleations are stopped. Then the final time t_n is determined from

$$1 \approx \int_{-\infty}^{t_n} dt \, \frac{4\pi v^3 (t_n - t)^3}{3} \, p_0 \, e^{-\hat{S}_E(t)}$$

$$\approx \frac{4\pi v^3 p_0}{3} e^{-\hat{S}_E(t_n)} \int_{-\infty}^{t_n} dt \, (t_n - t)^3 e^{-|\hat{S}_E'(t_n)|(t_n - t)}$$

$$x = t_n - t \qquad \int_0^\infty dx\, x^3 e^{-|\hat{S}_E'(t_n)| x} = \frac{3!}{|\hat{S}_E'(t_n)|^4}$$

$$\stackrel{\downarrow}{=} \frac{8\pi v^3 p_0}{|\hat{S}_E'(t_n)|^4} e^{-\hat{S}_E(t_n)}. \qquad (*)$$

By leaving out the volume factor, we can estimate the average „inverse volume" of the region that is in the broken phase:

$$\frac{1}{V} \approx \int_{-\infty}^{t_n} dt \, p_0 \, e^{-\hat{S}_E(t)} \approx \frac{p_0}{|\hat{S}_E'(t_n)|} e^{-\hat{S}_E(t_n)} \underset{(*)}{\overset{\uparrow}{=}} \frac{|\hat{S}_E'(t_n)|^3}{8\pi v^3}.$$

This is expressed through a distance scale $\ell := \left(\frac{1}{V}\right)^{-1/3}$.

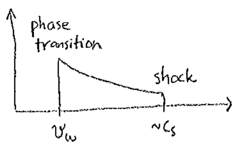
Finally, the „duration" of the transition is

$$\Delta t := \frac{\ell}{v} \sim \frac{1}{|\hat{S}_E'(t_n)|}. \qquad (**)$$

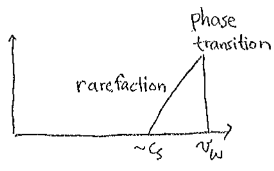
From p.3 we recall $\gamma = \frac{dx}{dt} = 3c_s^2 H$, so we can write

$$|\hat{S}_E'(t_n)| = 3 c_s^2 H \, |\hat{S}_E'(x_n)| \quad , \quad x_n = \ln\left(\frac{T_{max}}{T_n}\right). \qquad (***)$$

Summary: The nucleation temperature can be estimated from $(*)$, the distance/time scale from $(**)$. But these are just order-of-magnitude estimates. For instance, the hydrodynamics of bubble growth (related to v) is a complicated problem of its own.



„deflagration"



„detonation"

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Consequences:

The reason that cosmological phase transitions have been much studied is that they could leave behind „relics", such as a baryon asymmetry (cf. p. 41 ff) or gravitational waves.

Gravitational waves never equilibrate, so they are a „freeze-in" relic: their production is suppressed by $\sim 1/m_{Pl}^2$. In fact, the production rate of the energy density carried by momentum mode k can be written much like on p. 23, just weighting with ω_k:

$$\frac{de_{GW}}{dt\, d^3\vec{k}} = \frac{2}{\pi^2 m_{Pl}^2} \int_{\mathcal{X}} e^{ik(t-z)} \left\langle \hat{T}^{xy}(0)\, \hat{T}^{xy}(\mathcal{X}) \right\rangle .$$

To estimate this from phase transitions, one needs to express $T^{\mu\nu}$ in terms of plasma variables (e, p, T, and the local flow velocity u^μ):

$$T^{\mu\nu} = (e+p)u^\mu u^\nu - p g^{\mu\nu} + \phi^{,\mu}\phi^{,\nu} - \frac{g^{\mu\nu}\phi_{,\alpha}\phi^{,\alpha}}{2} ,$$

$$p = p_{fluid}(T) - V(\phi, T) , \quad e + p := T\partial_T p .$$

Extensive simulations suggest that the most important source of gravitational wave production are hydrodynamic sound waves, i.e. the enormous noise produced by the expanding and colliding bubbles.

Let us estimate the current wave length of waves produced. The homogeneity of the plasma is broken by the bubble distance scale ℓ (cf. p. 39), and the wave lengths should be less than this:

$$\lambda(t_0) \le \ell(t_n)\, \frac{a(t_0)}{a(t_n)}$$

time today

nucleation time

redshift by expansion

(**, ***) p. 39

$$\sim \frac{v}{3c_s^2 H |\hat{S}_E'(x_n)|} \left[\frac{s(T_n)}{s(T_0)} \underbrace{\frac{s(T_0)a^3(t_0)}{s(T_n)a^3(t_n)}}_{=1} \right]^{1/3}$$

$3c_s^2 \sim 1$, $H \sim T^2/m_{Pl}$

$$\sim \frac{v\, m_{Pl}}{T_n^2 |\hat{S}_E'(x_n)|} \cdot \frac{T_n}{T_0} \cdot \left[\frac{g_*(T_n)}{g_*(T_0)} \right]^{1/3}$$

T_0

$$\sim \frac{v}{|\hat{S}_E'(x_n)|} \cdot \underbrace{\frac{10^{19}\,\text{GeV}}{10^2\,\text{GeV}\; 10^{-3}\,\text{eV}}}_{\frac{10^{29}}{\text{GeV}\cdot\text{fm}}\cdot \text{fm} \,\sim\, 10^{13}\,\text{m}} \underbrace{\left(\frac{100}{2}\right)^{1/3}}_{3}$$

T_n

If the velocity is $\sim c_s \simeq 0.5$ (p. 2) and $|\hat{S}_E'(x_n)| \sim 100$, we get $\lambda(t_0) \lesssim 10^{11}$ m.

We expect a launch in ~2034 of a laser interferometer LISA with arm length $\sim 3\times 10^6$ km $= 3\times 10^9$ m, which is ideal for looking for this type of gravitational wave lengths.