Thermal phase transitions. 2. Examples

(1) How can we reliably work out what happens in the Standard Model?

Starting point: A naive 1-loop formula according to p. 31 states that

$$V^{(0)}_{\rm eff} + V^{(1)}_{\rm eff} = -\frac{m^2\bar\phi^2}{2} + \frac{\lambda\bar\phi^4}{4} + \sum_{\rm bosons}\int_{\vec k}\left[\frac{\epsilon_k}{2} + T\ln\left(1-e^{-\beta\epsilon_k}\right)\right]_{\epsilon_k=\sqrt{k^2+m^2_{\rm eff}}}$$
$$+ \sum_{\rm fermions}\int_{\vec k}\left[-\frac{\epsilon_k}{2} - T\ln\left(1+e^{-\beta\epsilon_k}\right)\right]_{\epsilon_k=\sqrt{k^2+m^2_{\rm eff}}}$$

But there are potential problems with this approximation:

* perturbation theory may not be trustworthy, because the expansion parameter may be $\mathcal{O}(1)$, cf. p. 32
* in fact, recalling $m^2_{{\rm eff},\phi} = -m^2 + 3\lambda\bar\phi^2$, ϵ_k becomes complex for $\bar\phi^2 < \frac{m^2}{3\lambda}$, and $V^{(1)}_{\rm eff}$ is ill-defined in the symmetric phase

A reliable way to address the transition is to use perturbation theory only for modes with $\omega_n \neq 0$, because $\omega_n^2 + m^2_{\rm eff}$ stays large and positive. Integrating out these modes, we may construct an effective field theory for the zero modes:*

$$S_{\rm eff} = \frac{1}{T}\int_V d^3x\left\{\frac{1}{2}(\partial_i\phi)^2 + \frac{1}{2}m_3^2\phi^2 + \frac{1}{4}\lambda_3\phi^4 + \dots\right\}.$$

from $\int_0^\beta d\tau$

only spatial derivatives, for $\partial_\tau\phi = 0$

$m_3^2 \approx -m^2 + \frac{\lambda T^2}{4}$

truncated higher-dimensional operators

* P. Ginsparg, „First and second order phase transitions in gauge theories at finite temperature", Nucl. Phys. B 170 (1980) 388; T. Appelquist and R.D. Pisarski, „High-temperature Yang-Mills theories and three-dimensional Quantum Chromodynamics", Phys. Rev. D 23 (1981) 2305

Yang-Mills: The identification of the low-energy degrees of freedom is most subtle for the gauge sector, so consider $L_E := \frac{1}{4}F^a_{\mu\nu}F^a_{\mu\nu}$.

We can write $F^a_{i0} = \mathcal{D}^{ab}_i A^b_0 - \partial_0 A^a_i$, where $\mathcal{D}^{ab}_i := \delta^{ab}\partial_i - gf^{abc}A^c_i$ is a covariant derivative in the adjoint representation. For static modes, $\partial_0 A^a_i = 0$, so

$$L_E^{(n=0)} \supset \frac{1}{4}F^a_{ij}F^a_{ij} + \frac{1}{2}(\mathcal{D}^{ab}_i A^b_0)(\mathcal{D}^{ac}_i A^c_0).$$

Consider now gauge transformations:

$$A'_\mu = UA_\mu U^{-1} + \frac{i}{g}U\partial_\mu U^{-1}, \qquad A_\mu := A^a_\mu T^a.$$

generators

Restricting to static fields, U should not depend on τ, so that we remain within the static subset.* Thus the effective theory is invariant under the reduced symmetry

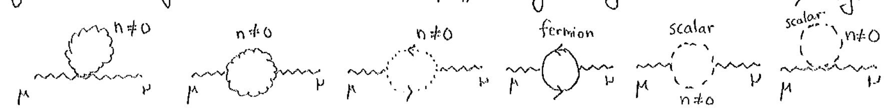
$$\begin{cases} A'_i = UA_iU^{-1} + \frac{i}{g}U\partial_iU^{-1}, \\ A'_0 = UA_0U^{-1}. \end{cases}$$

* Having U τ-independent is sufficient for this, but is it also necessary? Good question!

In other words, A_0 is a scalar field in the adjoint representation. This allows us to postulate the general form of the effective theory:

$$L_E^{(n=0)} = \frac{1}{4}F^a_{ij}F^a_{ij} + \frac{1}{2}(\mathcal{D}^{ab}_iA^b_0)(\mathcal{D}^{ac}_iA^c_0) + m_E^2\,{\rm Tr}(A_0^2) + \lambda_E^{(1)}\left({\rm Tr}[A_0^2]\right)^2 + \lambda_E^{(2)}\,{\rm Tr}(A_0^4) + \dots$$

Matching: The parameters of the effective theory can be determined by „matching Green's functions" / „integrating out $\omega_n \neq 0$", e.g.



$(\mu\nu) = (ij)$: gauge invariance guarantees that the result is $\mathcal{O}(\vec{k}^2)$, yielding a correction to the effective gauge coupling

$(\mu\nu) = (00)$: result can remain non-zero as $\vec{k} \to \vec{0}$, yielding $m_E^2 \neq 0$.

For illustration, this is how we compute a typical 1-loop diagram:

$$T \sum_{\omega_n \neq 0} \int \frac{d^d\vec{k}}{(2\pi)^d} \frac{1}{\omega_n^2 + k^2} = \int \frac{d^d\vec{k}}{(2\pi)^d} \frac{1}{k} \left[\frac{1}{2} + \frac{1}{e^{k/T} - 1} - \frac{T}{k} \right]$$

subtract $n=0$

like on page 30, with $m=0$

$$\frac{d^d\vec{k}}{(2\pi)^d} = \frac{2\, dk\, k^{d-1}}{(4\pi)^{d/2}\Gamma(d/2)} \; ; \qquad \frac{1}{e^{k/T}-1} = \frac{e^{-k/T}}{1 - e^{-k/T}} = \sum_{n=1}^{\infty} e^{-\frac{kn}{T}}$$

$$= \sum_{n=1}^{\infty} \frac{2}{(4\pi)^{d/2}\Gamma(d/2)} \int_0^\infty dk\, k^{d-2}\, e^{-\frac{kn}{T}}$$

$$= \sum_{n=1}^{\infty} \frac{2\,T^{d-1}}{(4\pi)^{d/2}\Gamma(d/2)\, n^{d-1}} \underbrace{\int_0^\infty dx\, x^{d-2}\, e^{-x}}_{\Gamma(d-1)}$$

$$= \frac{2\,T^{d-1}\,\Gamma(d-1)\,\zeta(d-1)}{(4\pi)^{d/2}\,\Gamma(d/2)}$$

$d=3$

$$= \frac{2T^2\,\zeta(2)}{4\pi\sqrt{4\pi}\,\Gamma(3/2)} = \frac{2T^2\,\frac{\pi^2}{6}}{4\pi\, 2\sqrt{\pi}\,\frac{1}{2}\sqrt{\pi}} = \underline{\underline{\frac{T^2}{12}}} \, .$$

Summing together the graphs, one gets

$$m_E^2 = g^2 T^2 \left(\frac{2}{3} + \frac{n_G}{3} + \frac{n_S}{6} \right) + \mathcal{O}(g^4 T^2) \, , \qquad \begin{array}{l} n_G = 3 \text{ (fermions)} \\ n_S = 1 \text{ (scalars)} \end{array}$$

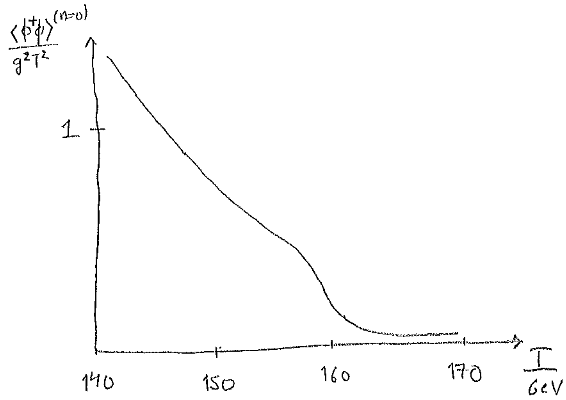
Bottom line: A_0 is „massive" because $m_E^2 > 0$, and can be integrated out! Only spatial („magnetic") gauge field components remain:

$$L_E^{(n=0)} = \underbrace{\frac{1}{4} F^a_{ij} F^a_{ij}}_{SU(2)} + \underbrace{\frac{1}{4} B_{ij} B_{ij}}_{U_Y(1)} + (D_i\phi)^\dagger (D_i\phi) + m_3^2 \phi^\dagger\phi + \lambda_3 (\phi^\dagger\phi)^2 + \ldots$$

$$\langle \phi^\dagger\phi \rangle_T = \frac{T^2}{6} + \langle \phi^\dagger\phi \rangle^{(n=0)}$$

from $\omega_n \neq 0$, cf. p. 30

The part $\langle \phi^\dagger\phi \rangle^{(n=0)}$ needs to be measured with lattice simulations because of the Linde problem (p. 30), and turns out to display a <u>smooth crossover</u>:



(ii) With additional fields, strong first-order transitions are possible

Prototype model: Consider a theory with two real scalar fields:

$$\mathcal{L}_M = \frac{1}{2}\partial^\mu\phi\partial_\mu\phi + \frac{1}{2}\partial^\mu\chi\partial_\mu\chi - V(\phi,\chi)\ ;\ V(\phi,\chi) = \frac{-m^2\phi^2}{2} + \frac{\lambda\phi^4}{4} - \frac{M^2\chi^2}{2} + \frac{\varkappa\chi^4}{4} + \frac{\gamma\phi^2\chi^2}{2}$$

$$L_E = -\mathcal{L}_M(it\to\tau)$$

$$L_E^{(n=0)} \supset \frac{1}{2}\partial_i\phi\partial_i\phi + \frac{1}{2}\partial_i\chi\partial_i\chi + V_3(\phi,\chi)\ ;\ V_3(\phi,\chi) = \frac{m_3^2\phi^2}{2} + \frac{\lambda_3\phi^4}{4} + \frac{M_3^2\chi^2}{2} + \frac{\varkappa_3\chi^4}{4} + \frac{\gamma_3\phi^2\chi^2}{2}$$

Thermal masses can be computed like on p. 34:

$$\overbrace{\phi\phi\ \frac{\lambda}{4}\underbrace{\phi\phi\phi\phi}_{3}}^{4} \Rightarrow m_3^2 \supset 3\lambda\frac{T^2}{12}$$

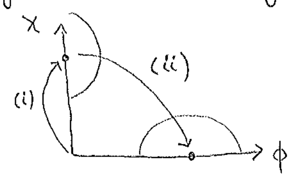
$$\overbrace{\phi\phi\ \frac{\gamma}{2}\phi\phi\chi\chi}^{2} \Rightarrow m_3^2 \supset \gamma\frac{T^2}{12}$$

And similarly for χ. So both symmetries tend to get restored:

$$m_3^2 \approx -m^2 + \frac{(3\lambda+\gamma)T^2}{12} \Rightarrow T_\phi^2 \approx \frac{12m^2}{3\lambda+\gamma}$$

$$M_3^2 \approx -M^2 + \frac{(3\varkappa+\gamma)T^2}{12} \Rightarrow T_\chi^2 \approx \frac{12M^2}{3\varkappa+\gamma}.$$

Two-stage transition: Suppose that the symmetry first breaks in the χ-direction ($T_\chi > T_\phi$), but that at low temperatures we return to ϕ_{min} ($V(\phi_{min},0) < V(0,\chi_{min})$). Then there could be a strong „tree-level" transition, from one deep valley to another through a saddle point:



Mapping the parameters: There are a number of constraints to satisfy, in order for step (ii) to take place at low temperatures ($T\approx 0$).

* where are the extrema?

$$\partial_\phi V = \partial_\chi V = 0 \Rightarrow \begin{cases} 0 = \phi(-m^2 + \lambda\phi^2 + \gamma\chi^2) \\ 0 = \chi(-M^2 + \varkappa\chi^2 + \gamma\phi^2) \end{cases}$$

So there are four possibilities:

(a) $\phi = \chi = 0$

(b) $\phi = 0,\quad \chi^2 = \frac{M^2}{\varkappa}$

(c) $\chi = 0,\quad \phi^2 = \frac{m^2}{\lambda}$

(d) $$\begin{pmatrix}\phi^2\\ \chi^2\end{pmatrix} = \frac{1}{\lambda\varkappa - \gamma^2}\begin{pmatrix}\varkappa & -\gamma \\ -\gamma & \lambda\end{pmatrix}\begin{pmatrix}m^2\\ M^2\end{pmatrix}.$$

* are the extrema stable?

$$\begin{pmatrix} \partial_\phi^2 V & \partial_\phi \partial_\chi V \\ \partial_\chi \partial_\phi V & \partial_\chi^2 V \end{pmatrix} = \begin{pmatrix} -m^2 + 3\lambda\phi^2 + \gamma\chi^2 & 2\gamma\phi\chi \\ 2\gamma\phi\chi & -M^2 + 3\varkappa\chi^2 + \gamma\phi^2 \end{pmatrix}$$

Insert here the extrema from p. 35 (all four cases):

(a): both directions unstable (maximum)

(b): both directions stable if $\frac{M^2}{m^2} > \frac{\varkappa}{\gamma}$

(c): both directions stable if $\frac{M^2}{m^2} < \frac{\gamma}{\lambda}$

(d): Saddle point** $\left(\frac{\varkappa}{\gamma} < \frac{\gamma}{\lambda} \text{ implies } \gamma^2 > \varkappa\lambda \right)$

** $\mathrm{Tr}(\ ) = \dfrac{2\varkappa\overbrace{(\gamma m^2 - \lambda M^2)}^{>0} + 2\lambda\overbrace{(\gamma M^2 - \varkappa m^2)}^{>0}}{\gamma^2 - \varkappa\lambda}$

$\mathrm{Det}(\ ) = \dfrac{-4\overbrace{(\gamma m^2 - \lambda M^2)}^{>0}\overbrace{(\gamma M^2 - \varkappa m^2)}^{>0}}{\gamma^2 - \varkappa\lambda}$

* Values of the potential at the stable minima:

(b): $V(0, \chi_{min}) = -\frac{M^4}{4\varkappa}$

(c): $V(\phi_{min}, 0) = -\frac{m^4}{4\lambda}$

$\Rightarrow$ the desired minimum is realized if $\frac{M^2}{m^2} < \sqrt{\frac{\varkappa}{\lambda}}$.

Putting together (denoting $\hat{\varkappa} := \frac{\varkappa}{\gamma}$, $\hat{\lambda} := \frac{\lambda}{\gamma}$), we need:

(i) $\frac{M^2}{m^2} > \frac{3\hat{\varkappa}+1}{3\hat{\lambda}+1}$ $\quad [T_\chi^2 > T_\phi^2]$

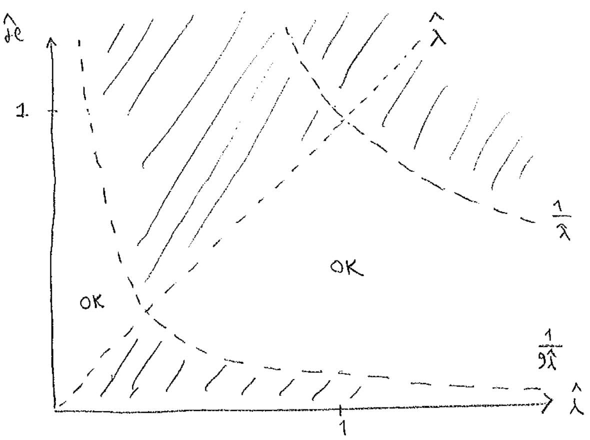
(ii) $\frac{1}{\hat{\lambda}} > \frac{M^2}{m^2} > \hat{\varkappa}$ $\quad$ [(b) & (c) stable]

(iii) $\sqrt{\frac{\hat{\varkappa}}{\hat{\lambda}}} > \frac{M^2}{m^2}$ $\quad$ [correct minimum at low T]

These can indeed be satisfied simultaneously***:

*** Eliminating $\frac{M^2}{m^2}$, the constraints can be combined into

$$\begin{cases} \hat{\varkappa} < \frac{1}{\hat{\lambda}} \\ (\hat{\varkappa} - \hat{\lambda})(\hat{\varkappa} - \frac{1}{9\hat{\lambda}}) < 0 \end{cases}$$



Summary:

In principle first order phase transitions are possible in extensions of the Standard Model.

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