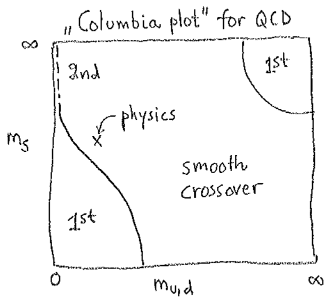
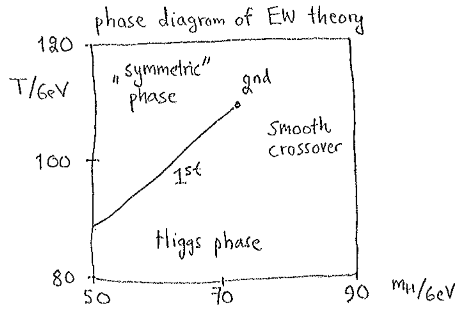
Thermal phase transitions. 1. Theoretical tools

Overall picture: Both QCD and the electroweak (EW) theory display a „broken symmetry" in vacuum, which gets „smoothly restored" at high temperatures:





In extensions of the electroweak theory, things might go otherwise. Given a model, we may ask:

(i) is there an actual phase transition?

(ii) what are its „equilibrium properties" (T_c, L, σ)? — critical temperature, latent heat, surface tension

(iii) how does it proceed in real time?

(iv) does it leave remnants (gravitational waves, baryon asymmetry)?

Quantum statistical physics: Let us define and compute a number of basic objects:

* partition function: $Z = \mathrm{Tr}\,(e^{-\beta \hat{H}})$, $\beta := \frac{1}{T}$, $\hat{H}$ = Hamiltonian

* free energy: $Z = e^{-\beta F} \Leftrightarrow F = -T \ln Z$.

* free energy density: $f = \frac{F}{V}$, V = volume $\to \infty$.

We compute these for a harmonic oscillator:

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{m\omega^2 \hat{x}^2}{2} = \hbar\omega\left(\hat{a}^\dagger \hat{a} + \frac{1}{2}\right).$$

Denote $\varepsilon := \hbar\omega$. Then

$$Z = \sum_{n=0}^{\infty} \langle n | e^{-\beta \hat{H}} | n \rangle = \sum_{n=0}^{\infty} e^{-\beta\varepsilon\left(n+\frac{1}{2}\right)} = \frac{e^{-\beta\varepsilon/2}}{1-e^{-\beta\varepsilon}},$$

$$F = \frac{\varepsilon}{2} + T \ln\left(1 - e^{-\beta\varepsilon}\right).$$

We also need the analogue of a „propagator". In order to be able to use this later on in field theory, let us set $m, \hbar \to 1$. Then the coefficient of $\hat{x}^2$ in $\hat{H}$ is $\frac{\varepsilon^2}{2}$, and

$$\langle \hat{x}^2 \rangle := \frac{\mathrm{Tr}\,(\hat{x}^2 e^{-\beta\hat{H}})}{Z} = -2T \frac{\partial}{\partial \varepsilon^2} \ln Z$$

$$= \frac{1}{\varepsilon}\frac{\partial F}{\partial \varepsilon} = \frac{1}{\varepsilon}\left[\frac{1}{2} + \frac{e^{-\beta\varepsilon}}{1-e^{-\beta\varepsilon}}\right]$$

$$= \frac{1}{\varepsilon}\left[\frac{1}{2} + \underbrace{\frac{1}{e^{\beta\varepsilon}-1}}_{n_B(\varepsilon)\ \text{(Bose distribution)}}\right]$$

Statistical field theory: We can take a short cut from the harmonic oscillator to (free) scalar field theory, simply by replacing $\varepsilon \to \varepsilon_k := \sqrt{k^2+m^2}$.

$$Z_\phi = \prod_{\vec k} \exp\left\{-\frac{1}{T}\left[\frac{\varepsilon_k}{2} + T\ln\left(1-e^{-\beta\varepsilon_k}\right)\right]\right\},$$

$$f_\phi = \lim_{V\to\infty}\frac{F_\phi}{V} = \underbrace{\lim_{V\to\infty}\frac{1}{V}\sum_{\vec k}}_{=\int_{\vec k}} \underbrace{\left[\frac{\varepsilon_k}{2} + T\ln\left(1-e^{-\beta\varepsilon_k}\right)\right]}_{\sim E-TS}$$

The thermal part $f_T(m) := T\int_{\vec k}\ln\left(1-e^{-\beta\varepsilon_k}\right)$ is exponentially convergent. It has a formal expansion* which is non-analytic in m^2:

$$f_T(m) = \underbrace{-\frac{\pi^2T^4}{90}}_{\text{like } -p \text{ on p. 2}} + \frac{m^2T^2}{24} - \frac{m^3T}{12\pi} - \frac{m^4}{32\pi^2}\left[\ln\left(\frac{m}{4\pi T}\right)+\gamma_E - \frac{3}{4}\right] + O\left(\frac{m^6}{\pi^4T^2}\right).$$

$m := \sqrt{m^2}$

* L. Dolan and R. Jackiw, „Symmetry behavior at finite temperature", Phys. Rev. D 9 (1974) 3320

Example: Higgs condensate We may use the above expansion to estimate $\langle\phi^\dagger\phi\rangle$ at high T:

$$\langle\phi^\dagger\phi\rangle \simeq \frac{1}{2}\cdot\sum_{\mu=0}^{3} \int_{\vec k}\underbrace{\frac{1}{\varepsilon_k}\left[\frac{1}{2}+n_B(\varepsilon_k)\right]}_{\text{p. 29: } 2\frac{\partial}{\partial\varepsilon_k^2}\left[\frac{\varepsilon_k}{2}+T\ln\left(1-e^{-\beta\varepsilon_k}\right)\right];\ \frac{\partial}{\partial\varepsilon_k^2}\overset{!}{=}\frac{\partial}{\partial m^2}}$$

$$\boxed{\phi = \frac{1}{\sqrt{2}}\begin{pmatrix}\phi_2+i\phi_1\\ \phi_0-i\phi_3\end{pmatrix}}$$

$$= (\text{vacuum part}) + \sum_{\mu=0}^{3}\frac{\partial}{\partial m^2}f_T(m) = \frac{T^2}{6} + O(mT).$$

Therefore, at $T \gg v \simeq 246$ GeV, zero-temperature masses $m_W^2 = \frac{g_w^2 v^2}{4}$ may get replaced with thermal masses $\sim g_w^2T^2$ (more later).

Issues of convergence: In perturbation theory, the most important domain is where propagators are largest:

$$\frac{1}{\varepsilon_k}\left[\frac{1}{2} + \frac{1}{e^{\varepsilon_k/T}-1}\right] \approx \frac{1}{\varepsilon_k}\left[\frac{1}{2}+\frac{1}{\varepsilon_k/T + \varepsilon_k^2/2T^2+\ldots}\right] = \frac{T}{\varepsilon_k^2}+O\left(\frac{1}{T}\right).$$

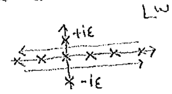
The large term at $\varepsilon_k \ll T$ originates from Bose-Einstein-enhancement. But there is also a very useful alternative interpretation**:

$$\frac{1}{\varepsilon_k}\left[\frac{1}{2}+\frac{1}{e^{\varepsilon_k/T}-1}\right] = T\sum_{\omega_n}\frac{1}{\omega_n^2+\varepsilon_k^2}, \quad \omega_n := 2\pi Tn,\ n\in\mathbb{Z}.$$

** This can be verified by considering the contour integral

$$\int \frac{d\omega}{2\pi i}\frac{1}{\omega^2+\varepsilon^2}\, i n_B(i\omega).$$

Poles:



Closing the contours in two different ways yields the desired equality.

Then the large term is associated with the „Matsubara zero mode" $\omega_n = 0$. With these ingredients, we can estimate a dimensionless loop expansion parameter (after integration over momenta, so that m is the only scale):

explicit factor seen above (T)
from vertices (g^2)
from loop integrals (π)
to make it dimensionless (m)

$$\frac{g^2T}{\pi m} \sim \begin{cases}\frac{g}{\pi^2}, & \text{if } m\gtrsim \pi T\\ \frac{g}{\pi}, & \text{if } m\sim gT\\ 1, & \text{if } m\sim \frac{g^2T}{\pi}\end{cases}$$

*** A.D. Linde, „Infrared problem in thermodynamics of Yang-Mills gas", Phys. Lett. B 96 (1980) 289

The breakdown of the loop expansion for very small masses is known as the „Linde problem".***

Imaginary-time formalism: It can be shown (replacing $e^{-\frac{i\hat{H}t}{\hbar}} \to e^{-\beta\hat{H}}$ in the usual derivation) that the partition function can be expressed as a „Euclidean"/„imaginary-time" path integral:

$$Z = \int_{b.c.} \mathcal{D}\phi \; e^{-S_E} ,$$

$$S_E = \int_0^\beta d\tau \int_V d^3\vec{x} \; L_E ,$$

$$L_E := -\mathcal{L}_M(it \to \tau) .$$

Here boundary conditions (b.c.) are periodic for bosons and antiperiodic for fermions. Check:

$$\phi(\beta,\vec{x}) = \phi(0,\vec{x}) \Rightarrow e^{i\omega_n\beta} = 1 \Rightarrow \omega_n = 2\pi T n \quad \text{ok!}$$

We apply this to a real scalar field with $V(\phi) = -\frac{m^2\phi^2}{2} + \frac{\lambda\phi^4}{4}$:

$$\mathcal{L}_M = \frac{1}{2}(\partial_t\phi)^2 - \frac{1}{2}(\nabla\phi)^2 - V(\phi)$$

$$\Rightarrow L_E = \frac{1}{2}(\partial_\tau\phi)^2 + \frac{1}{2}(\nabla\phi)^2 + V(\phi) .$$

Effective potential: We return to a finite volume for a moment. Let $\bar{\phi}$ be the mode with $\omega_n = 0$, $\vec{k} = \vec{0}$, and ϕ' the other modes. Note that $\int_0^\beta d\tau \int_V d^3\vec{x}\, \phi' = 0$! Now insert $\phi = \bar{\phi} + \phi'$ into Z:

$$Z = \int_{-\infty}^{\infty} d\bar{\phi} \int \mathcal{D}\phi' \; e^{-S_E[\phi = \bar{\phi}+\phi']} =: \int_{-\infty}^{\infty} d\bar{\phi} \; e^{-\frac{V}{T} V_{\text{eff}}(\bar{\phi})}$$

$$= \int_{-\infty}^{\infty} d\bar{\phi} \; e^{-\frac{V}{T}\left[V_{\text{eff}}(\bar{\phi}_{\min}) + \frac{1}{2}V''_{\text{eff}}(\bar{\phi}_{\min})(\bar{\phi}-\bar{\phi}_{\min})^2 + \dots\right]}$$

$$\Rightarrow f_\phi = V_{\text{eff}}(\bar{\phi}_{\min}) + \mathcal{O}\left(\frac{\ln V}{V}\right) .$$

Insert $\phi = \bar{\phi} + \phi'$ in L_E:

$$\frac{1}{2}(\partial_\mu\phi)^2 \to \qquad\qquad \frac{1}{2}(\partial_\mu\phi')^2$$

$$-\frac{1}{2}m^2\phi^2 \to -\frac{1}{2}m^2\bar{\phi}^2 - m^2\bar{\phi}\phi' \quad -\frac{1}{2}m^2\phi'^2$$

$$+\frac{1}{4}\lambda\phi^4 \to \underbrace{+\frac{1}{4}\lambda\bar{\phi}^4}_{\text{independent of } \tau,\,\vec{x}} \; \underbrace{+\lambda\bar{\phi}^3\phi'}_{\substack{\text{vanishes because} \\ \int d\tau\int d^3\vec{x}\,\phi'=0}} \; \underbrace{+\frac{3}{2}\lambda\bar{\phi}^2\phi'^2}_{\substack{\text{like free theory} \\ \text{but with mass} \\ m_{\text{eff}}^2 := -m^2+3\lambda\bar{\phi}^2}} \; \underbrace{+\lambda\bar{\phi}\phi'^3 + \frac{1}{4}\lambda\phi'^4}_{\text{interactions}}$$

In summary, the effective potential V_{eff} is composed of the following parts:

* tree-level potential: $V_{\text{eff}}^{(0)}(\bar{\phi}) = -\frac{1}{2}m^2\bar{\phi}^2 + \frac{1}{4}\lambda\bar{\phi}^4$

* 1-loop potential: like on p.30 but now with m_{eff}:

$$V_{\text{eff}}^{(1)}(\bar{\phi}) = \int_{\vec{k}} \left[\frac{\varepsilon_k}{2} + T\ln\left(1 - e^{-\beta\varepsilon_k}\right)\right] ,$$

$$\varepsilon_k = \sqrt{k^2 + m_{\text{eff}}^2}$$

* higher order corrections from the interaction part.

Phase transition:

Let us insert the high-temperature expansion from p.30 and see what kind of an effect $V^{(1)}_{eff}$ has:

* leading term $-\frac{\pi^2 T^4}{90}$ $\Rightarrow$ independent of $\bar\phi$ $\Rightarrow$ no effect

* NLO term $\frac{m_{eff}^2 T^2}{24} = \frac{-m^2 + 3\lambda\bar\phi^2}{24} T^2$

$$\Rightarrow V^{(0)}_{eff} + V^{(1)}_{eff} \approx [\bar\phi\text{-indep.}] + \frac{1}{2}\Big(-m^2 + \frac{\lambda T^2}{4}\Big)\bar\phi^2 + \frac{1}{4}\lambda\bar\phi^4$$

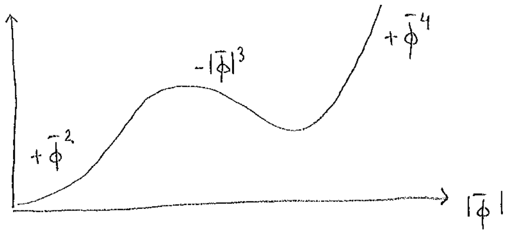
„thermal mass correction"

Thus the symmetry seems to get restored at $T_c \approx \frac{2m}{\sqrt\lambda}$.

* NNLO term $-\frac{m_{eff}^3 T}{12\pi} = -\frac{T}{12\pi}\left(-m^2 + 3\lambda\bar\phi^2\right)^{3/2}$.

Let us for simplicity put $m^2 \to 0$

$$\Rightarrow V^{(0)}_{eff} + V^{(1)}_{eff} \approx [\bar\phi\text{-indep.}] + \frac{\lambda}{8}T^2\bar\phi^2 - \frac{T}{12\pi}(3\lambda)^{3/2}|\bar\phi|^3 + \frac{\lambda}{4}\bar\phi^4$$



This looks a bit like a „fluctuation induced" / „radiatively generated" first order phase transition!

However at the end we have to ask <u>whether such a prediction is reliable?</u> Rough estimate:

* „broken minimum" is where $\frac{T}{\pi}\lambda^{3/2}|\bar\phi|^3 \sim \lambda\bar\phi^4$

$$\Rightarrow |\bar\phi| \sim \lambda^{1/2}\frac{T}{\pi}.$$

* dimensionless loop expansion parameter (p.30) is

$$\frac{\lambda T}{\pi m_{eff}} \sim \frac{\lambda T}{\pi\sqrt{\lambda\bar\phi^2}} \sim \frac{\lambda^{1/2} T}{\pi|\bar\phi|} \sim 1.$$

So we conclude that the prediction for a <u>first-order</u> transition is <u>not</u> reliable (series does not converge). In fact, in scalar field theory the transition is of 2nd order (continuous).

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