Freeze-in dark matter. 1. Basic picture

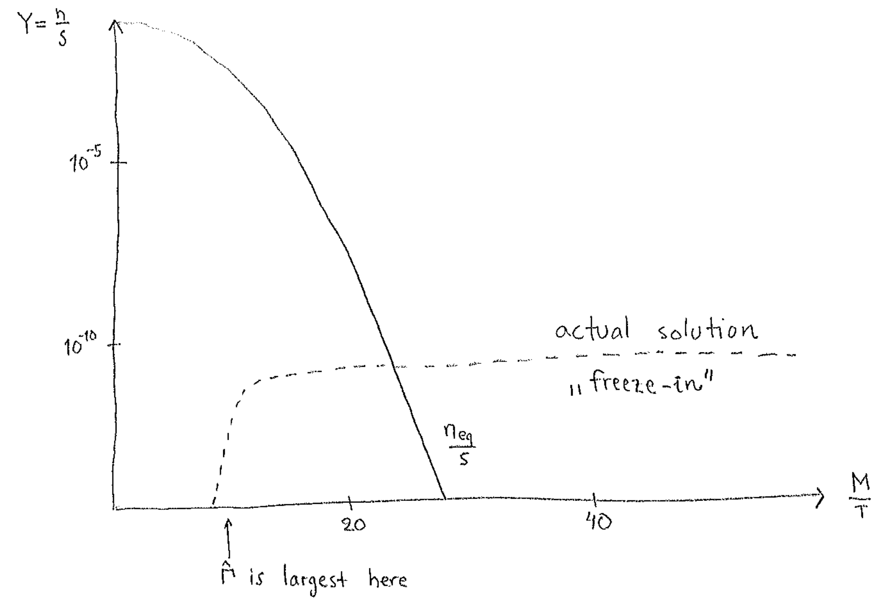
Let us return back to the master equation from p. 6/14:

$$\frac{df(x,k_T)}{dx} \approx -\hat{\Gamma}(k_T)\left[f(x,k_T) - f_{eq}(\omega_T)\right], \quad k_T = k(T_0)\frac{a(T_0)}{a(T)}, \quad \hat{\Gamma} = \frac{\Gamma}{\mathcal{J}}, \quad f_{eq} \in \{n_F, n_B\}.$$

Now the idea is that if $\hat{\Gamma} \ll 1$ all the time, and we start with the initial condition $f(0,k_T) = 0$, then $f(x,k_T) \ll f_{eq}(\omega_T)$ for a while. Then the equation reads

$$\frac{df(x,k_T)}{dx} \approx \hat{\Gamma}(k_T)\, f_{eq}(\omega_T). \qquad (*)$$

We see that the right-hand side is positive; a certain density of the dark matter particles is certainly produced. This mechanism could be interesting if $\hat{\Gamma}$ peaks in some temperature range, so that the solution is insensitive to the starting time. Sketch (to be contrasted with p.16):



Before going to a concrete example, we take the opportunity to illustrate how eq. (*) can be derived from Quantum Field Theory.

Formal derivation: (sketch)

Consider a complex scalar field ϕ, coupled to some gauge-invariant combination G of Standard Model fields through a very small coupling h, $|h| \ll 1$:

$$\mathcal{L} = \mathcal{L}_{SM} + \partial_\mu \phi^* \partial^\mu \phi - M^2 \phi^* \phi - (h\phi^* G + h^* G^* \phi).$$

In order to derive a rate equation, we return to the canonical formalism. Then the interactions between ϕ and the Standard Model are contained in

$$\hat{H}_{int} = \int_{\vec{x}} \left(h\hat{\phi}^\dagger \hat{G} + h^* \hat{G}^\dagger \hat{\phi}\right).$$

Like in the derivation of Fermi's Golden Rule, we now go to the <u>interaction</u> (or Dirac) <u>picture</u>. The interaction Hamiltonian in the interaction picture reads

$$\hat{H}_I = e^{i\hat{H}_0 t}\, \hat{H}_{int}\, e^{-i\hat{H}_0 t} ,$$

where $\hat{H}_0$ includes the free motion of ϕ and all SM dynamics. In this picture, the field operator has the plane-wave form,

$$\hat{\phi}_I = \int \frac{d^3\vec{p}}{\sqrt{(2\pi)^3 2\omega_p}} \left(\hat{a}_p e^{-iP\cdot X} + \hat{b}_p^\dagger e^{iP\cdot X} \right),$$

where $P := (\omega_p, \vec{p})$; $\omega_p = \sqrt{p^2 + M^2}$; $X = (t, \vec{x})$. Therefore

$$\hat{H}_I = \int_{\vec{x}} \int \frac{d^3\vec{p}}{\sqrt{(2\pi)^3 2\omega_p}} \Big\{ \big[h\, \hat{a}_p^\dagger \hat{C}_I + h^* \hat{C}_I^\dagger \hat{b}_p^\dagger \big](X)\, e^{iP\cdot X} + \big[h\, \hat{b}_p \hat{C}_I + h^* \hat{C}_I^\dagger \hat{a}_p \big](X)\, e^{-iP\cdot X} \Big\} .$$

Let $|\vec{k}\rangle := \hat{a}_k^\dagger |0\rangle$ be a state with one ϕ-particle of momentum $\vec{k}$. Consider an initial state $|I\rangle$ and a final state $|F\rangle$, with

$$|I\rangle := |i\rangle \otimes |0\rangle , \quad |F\rangle := |f\rangle \otimes |\vec{k}\rangle ,$$

where $|i\rangle$ and $|f\rangle$ are states in the Hilbert space of the heat bath. To first order in h, the transition matrix element reads

$$T_{FI} = \langle F | \int_0^t dt'\, \hat{H}_I(t') | I \rangle .$$

The thermally averaged particle production rate can be obtained as

$$\frac{\dot{f}}{(2\pi)^3} = \lim_{t, V \to \infty} \sum_{f, i} \frac{e^{-E_i/T}}{Z} \frac{|T_{FI}|^2}{tV} ,$$

where the thermal average is over <u>initial states</u> (final states are unconstrained), $Z \equiv \sum_i e^{-E_i/T}$ is the partition function, and $(2\pi)^3$ is related to the normalization of f (such that $\frac{N}{V} = \int \frac{d^3\vec{k}}{(2\pi)^3} f$).

Making use of

$$\langle \vec{k} | \hat{H}_I | 0 \rangle = \langle 0 | \hat{a}_k \hat{H}_I | 0 \rangle = \langle 0 | [\hat{a}_k, \hat{H}_I] | 0 \rangle$$

and $[\hat{a}_k, \hat{a}_p^\dagger] = \delta^{(3)}(\vec{p} - \vec{k})$, we find

$$T_{FI} = \int_0^t dt' \int_{\vec{x}} \frac{e^{iK\cdot X'}}{\sqrt{(2\pi)^3 2\omega_k}}\, h\, \langle f | \hat{C}_I(X') | i \rangle ,$$

where $X' = (t', \vec{x})$. Correspondingly,

$$|T_{FI}|^2 = \frac{|h|^2}{(2\pi)^3 2\omega_k} \int_{X', Y'} e^{iK\cdot(X' - Y')} \langle i | \hat{C}_I^\dagger(Y') | f \rangle \langle f | \hat{C}_I(X') | i \rangle .$$

Now we can make use of $\sum_f |f\rangle\langle f| = \mathbb{1}$. Moreover,

$$\frac{1}{Z} \sum_i e^{-E_i/T} \langle i | \cdots | i \rangle = \frac{\mathrm{Tr}\,[e^{-\hat{H}/T}(\cdots)]}{\mathrm{Tr}\,[e^{-\hat{H}/T}]} =: \langle \cdots \rangle .$$

$\rightarrow$

This yields

$$\dot{f} = \lim_{t,V\to\infty} \frac{|h|^2}{2\omega_k} \cdot \frac{1}{tV} \cdot \int_{X',Y'} e^{iK\cdot(X'-Y')} \langle \hat{C}^+(Y')\hat{C}(X')\rangle ,$$

where the subscript „I" could be eliminated, since expectation values are picture-independent.

Now we make use of translational invariance:

$$\langle \hat{C}^+(Y')\hat{C}(X')\rangle = \phi(X'-Y')$$

$$\Rightarrow \int_{X',Y'} e^{iK\cdot(X'-Y')}\phi(X'-Y') \underset{\boxed{X'\to X'+Y'}}{=} \underbrace{\int_{Y'}}_{tV} \int_{X'} e^{iK\cdot X'} \underbrace{\langle \hat{C}^+(0)\hat{C}(X')\rangle}_{\phi(X')}$$

$$\Rightarrow \dot{f} = \frac{|h|^2}{2\omega_k} \int_{X'} e^{iK\cdot X'} \langle \hat{C}^+(0)\hat{C}(X')\rangle . \qquad (**)$$

The correlator here involves a particular time ordering (it's not the usual „time-ordered" one), and is called a Wightman function, denoted by $\Pi_<(K)$.

In thermal field theory, an important time ordering is the so-called retarded correlator:

$$\Pi_R(K) := i\int_{X'} e^{iK\cdot X'}\langle [\hat{C}(X'), \hat{C}^+(0)]\,\Theta(t')\rangle .$$

It can be shown* that

$$\Pi_<(K) = 2n_B(\omega_k)\,\mathrm{Im}\,\Pi_R(K) .$$

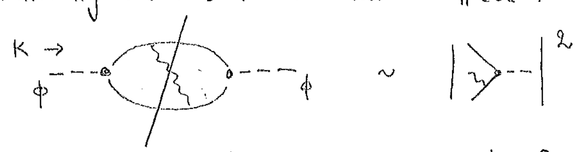
* To do this, evaluate the expectation values in the energy eigenbasis, and carry out the Fourier transforms.

Inserting this into (**) and comparing with eq. (*) from p. 21 [with $f_{eq} = n_B$], we finally identify

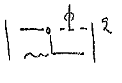
$$\boxed{\Gamma(k) = \frac{|h|^2}{\omega_k}\,\mathrm{Im}\,\Pi_R(K) .}$$

Interpretation:

(i) The operator $\hat{C}$ contains a product of fields. The imaginary part means considering „amplitudes squared" (cf. optical theorem). Diagrammatically we illustrate this with a „cut":



(ii) In vacuum, such a cut can only originate from decays or inverse decays, but in thermal field theory the cut lines can be attached either to the initial or final state, if they are properly „dressed" with Bose or Fermi distributions:

$\sim n_F n_F(1+n_B)$  $+ \; n_F n_B(1-n_F)$ 

(iii) The factor $f_{eq} = n_B$ in eq. (*) on p. 21 gives the probability that the on-shell energy ω can be extracted from a thermal fluctuation.

Non-equilibrium expansion: Q: if some particle is out of equilibrium, aren't the steps on p. 1-2, which assumed equilibrium, invalidated?

Let us assume that the plasma has two components:

$$\underbrace{e = e_T}_{\text{thermal part}} + \underbrace{e_\phi}_{\text{non-thermal part}}\ , \quad p = p_T + p_\phi \ ; \quad de_T = T ds_T\ , \ e_T + p_T = T s_T\ .$$

We rewrite eq. (F3) from p.1 as $0 = \dot e a^3 + 3(e+p)\dot a a^2 = \partial_t(e a^3) + p\partial_t a^3$ and use $(\partial_t e_T)a^3 + (e_T+p_T)\partial_t a^3 = T\partial_t(s_T a^3)$ for the thermal part:

$$T\partial_t(s_T a^3) = -\partial_t(e_\phi a^3) - p_\phi\partial_t(a^3)\ .$$

Now write e_ϕ like on p. 2:

$$a^3(t)\, e_\phi = a^3(t)\int_{\vec k_t}\omega_t\, f(t,k_t) \overset{k_t = \frac{k_0 a(t_0)}{a(t)}}{=} a^3(t_0)\int_{\vec k_0}\sqrt{M^2 + \frac{k_0^2 a^2(t_0)}{a^2(t)}}\, f(t,k_t)$$

Then the time derivative has two terms. Acting on ω_t, we get

$$-\partial_t\sqrt{M^2+\frac{k_0^2a^2(t_0)}{a^2(t)}} = \frac{-1}{2}\cdot\frac{1}{\omega_t}\cdot\left(-\frac{k_0^2a^2(t_0)}{a^3(t)}\right)2\dot a(t) = +\frac{k_t^2 H}{\omega_t}\ .$$

This cancels against the term from $-p_\phi\partial_t(a^3)$:

$$-p_\phi\, 3a^3(t)H = -\int_{\vec k_t}\frac{k_t^2}{3\omega_t}f(t,k_t)\cdot 3a^3(t)H \overset{k_t = \frac{k_0 a(t_0)}{a(t)}}{=} -a^3(t_0)\int_{\vec k_0}\frac{k_t^2 H}{\omega_t}f(t,k_t)\ .$$

So the only term left over reads

$$T\partial_t(s_T a^3) = -a^3(t)\int_{\vec k_t}\omega_t\,\partial_t f(t,k_t) \qquad (***)$$

$$\overset{p.5}{=} a^3(t)\int_{\vec k_t}\omega_t\,\Gamma(k_t)\left[f(t,k_t) - f_{eq}\right]$$

$$\Longleftrightarrow \quad \partial_x(s_T a^3) = a^3\int_{\vec k_T}\frac{\omega_T}{T}\hat\Gamma(k_T)\left[f(x,k_T) - f_{eq}\right]\ .$$

Interpretation: Entropy conservation is violated if two conditions are satisfied simultaneously:

(i) $\hat\Gamma$ is substantial

(ii) $f(x,k_T) \neq f_{eq}$

However, these requirements go against each other (large $\hat\Gamma$ drives f to equilibrium), so the effect can „normally" be neglected.

Physically, eq. (***) is related to the thermodynamic relation $TdS = dE$, with $S = sV \approx s a^3$. This explains why the energy release $\sim\omega_t\partial_t f$ appears on the right-hand side.