Expanding background and overall picture

As a starting point, we recall from general relativity that the homogeneous and isotropic (Friedmann-Lemaître-) Robertson-Walker metric* $ds^2 = -dt^2 + a^2(t)\,d\vec{r}_k^{\,2}$ and the „ideal" energy-momentum tensor $T^\mu{}_\nu = \mathrm{diag}(-e,p,p,p)$, where e = energy density, p = pressure, lead to the so-called Friedmann equations:

*later on we use the „particle-physics" metric convention $ds^2 = dt^2 - a^2(t)\,d\vec{r}_k^{\,2}$.

$$\frac{\dot a^2}{a^2} + \frac{k}{a^2} = \frac{8\pi G_N}{3}\, e \qquad (F1)$$

$$-\frac{\ddot a}{a} = \frac{4\pi G_N}{3}(e+3p) \qquad (F2)$$

$$\dot e = -3(e+p)\frac{\dot a}{a} \qquad (F3)$$

Comments:

* often the equations contain a „cosmological constant" Λ, but we have included it in e, p without loss of generality
* in the literature, e is often denoted by ρ
* we write the Newton constant as $G_N \equiv \frac{1}{m_{Pl}^2}$, where $m_{Pl} \approx 1.22\times 10^{19}$ GeV is called the Planck mass
* observations indicate that the universe is „flat", $k=0$, which we'll assume [this unnatural-looking „flatness problem" is explained by inflation (cf. later), which guarantees that $\frac{k}{a^2} \lll \frac{\dot a^2}{a^2}$]
* only two of the Friedmann equations are independent; we choose (F1)&(F3).

$$\left[\frac{d}{dt}F_1 \Rightarrow \frac{2\dot a}{a}\left\{\frac{\ddot a}{a} - \frac{\dot a^2}{a^2} - \frac{k}{a^2}\right\} = \frac{8\pi G_N}{3}\dot e \Rightarrow \frac{2\dot a}{a}\left\{-\frac{4\pi G_N}{3}(e+3p) - \frac{4\pi G_N}{3}(2e)\right\} = \frac{4\pi G_N}{3}(2\dot e) \Leftrightarrow -3\frac{\dot a}{a}(e+p) = \dot e \right]$$

Denoting the Hubble rate by $H \equiv \frac{\dot a}{a}$ we thus have

$$\begin{cases} H = \sqrt{\dfrac{8\pi e}{3 m_{Pl}^2}} & (F1') \\ \dot e = -3(e+p)H & (F3') \end{cases}$$

Let us furthermore assume that the matter filling the universe is in thermodynamic equilibrium and that chemical potentials are small ($\mu \ll T$). Then e and p are related to each other:

$$E = TS - pV + \cancel{\mu N} \qquad (T1)$$

$$dE = TdS - pdV + \cancel{\mu dN} \qquad (T2)$$

Now write $e \equiv \frac{E}{V}$ and $s \equiv \frac{S}{V}$ (entropy density). Dividing (T1) by V gives $e = Ts - p$. From (T2) we find

$$de = d\left(\frac{E}{V}\right) = \frac{dE}{V} - \frac{E}{V^2}dV = \frac{TdS}{V} - \cancel{\frac{pdV}{V}} - \frac{TS}{V^2}dV + \cancel{\frac{pdV}{V}} = Td\left(\frac{S}{V}\right) = Tds.$$

In addition, $de = dTs + Tds - dp \Rightarrow dp = s\,dT$. So we have relations characteristic of the fact that e and p are Legendre transforms of each other:

$$\begin{cases} e = Ts - p & (T1') \\ de = Tds & (T2') \\ dp = s\,dT & (T3') \end{cases}$$

↘

With the help of the thermodynamic relations, (F3') can be simplified:

$$(T2') \Rightarrow \quad \dot{e} = T\dot{s}$$

$$(T1') \Rightarrow \quad e + p = Ts$$

$$(F3'): \quad \cancel{T}\dot{s} = -3\cancel{T}s\frac{\dot{a}}{a} \quad \Leftrightarrow \quad \boxed{\frac{d}{dt}(sa^3) = 0} \qquad (F3'') .$$

This important relation is known as <u>entropy conservation</u>.

<u>Thermodynamic functions:</u> In order to solve the Friedmann equations, we need to know s and e as a function of T. Let us assume that the plasma consists of free particles, with masses m_i and degeneracies g_i. Denoting $\omega_i := \sqrt{k^2 + m_i^2}$, we then have

$$e = \sum_i g_i \int \frac{d^3\vec{k}}{(2\pi)^3}\, \omega_i f_i(\omega_i) \quad , \quad f_i := \frac{1}{e^{\omega_i/T} \pm 1} .$$

Recalling $\vec{v} = \nabla_{\vec{k}}\,\omega = \frac{\vec{k}}{\omega}$ and that pressure is „momentum flux", we can also write

$$p = \sum_i g_i \int_{\vec{k}} \frac{\vec{k}^2}{3\omega_i} f_i(\omega_i) \quad , \quad \int_{\vec{k}} := \int \frac{d^3\vec{k}}{(2\pi)^3} .$$

Entropy density is obtained from (T3') as $s = p'$ $(p' := dp/dT)$, and we will also need „speed of sound squared":

$$c_s^2 := \frac{p'}{e'} = \frac{s}{Ts'} .$$

For $m \ll T$: $\displaystyle\int_{\vec{k}} \frac{k}{e^{k/T} - 1} = \frac{\pi^2 T^4}{30}$, $\displaystyle\int_{\vec{k}} \frac{k}{e^{k/T} + 1} = \frac{7}{8} \cdot \frac{\pi^2 T^4}{30}$.

For $m \gg T$: $\displaystyle\int_{\vec{k}} \frac{\omega}{e^{\omega/T} \pm 1} \approx m \left(\frac{mT}{2\pi}\right)^{3/2} e^{-m/T}$.

Therefore $\displaystyle e \approx g_* \frac{\pi^2 T^4}{30}$, $\displaystyle g_* := \sum_{\substack{\text{bosons} \\ \text{with} \\ m_i \ll T}} g_i + \frac{7}{8} \sum_{\substack{\text{fermions} \\ \text{with} \\ m_i \ll T}} g_i$.

Given that $\frac{\vec{k}^2}{\omega} \approx k$ for small m, we similarly deduce

$$p \approx g_* \frac{\pi^2 T^4}{90} \quad , \quad s = g_* \frac{4}{3} \frac{\pi^2 T^3}{30} ,$$

$$c_s^2 \approx \frac{s}{Ts'} \approx \frac{T^3}{3T^3} = \frac{1}{3} .$$

Time vs. temperature

In particle cosmology, it is useful to track the evolution of the universe in terms of T rather than t, because T immediately characterizes the energy scale of the relevant processes.

So let us define $\boxed{x := \ln\left(\frac{T_{max}}{T}\right)}$, $T_{max} \overset{e.g.}{=} 10^{15}\,\text{GeV}$.

In astrophysics people rather use the redshift z, defined as

$$\frac{a(t_0)}{a(t)} = 1 + z ,$$

where t_0 = present day

In order to transcribe the Friedmann equations to depend on x, we need the Jacobian

$$J \equiv \frac{dx}{dt} = -\frac{\dot{T}}{T} .$$

$c_s^2 = \frac{s}{Ts'}$, $H = \frac{\dot a}{a}$

From (F3"): $$0 = \frac{\partial_t (sa^3)}{sa^3} = \frac{s'\dot{T}}{s} + \frac{3\dot{a}}{a} = -\frac{J}{c_s^2} + 3H$$

$$\Rightarrow \boxed{J = 3c_s^2 H} .$$

Given that $c_s^2 \approx \frac{1}{3}$, J is often approximated through H.

Now we can insert the value of H from (F1') and determine the relation of T and t:

$$J = -\frac{\dot{T}}{T} \overset{3c_s^2 = 1}{\approx} \sqrt{\frac{8\pi}{3}\frac{g_* \pi^2}{30}}\frac{T^2}{m_{pe}} = \frac{2}{3}\sqrt{\frac{g_* \pi^3}{5}}\frac{T^2}{m_{pe}}$$

if g_* is kept constant

$$\Leftrightarrow \quad -\frac{dT}{T^3}\,\frac{3}{2}\sqrt{\frac{5}{g_* \pi^3}}\,m_{pe} = dt$$

Here we note that $-\frac{dT}{T^3} = \frac{1}{2} d\left(\frac{1}{T^2}\right)$.

As an initial condition, let us formally set $t \to 0$ for $T \to \infty$. Then we immediately obtain

$$\boxed{t = \frac{3}{4}\sqrt{\frac{5}{g_* \pi^3}}\frac{m_{pe}}{T^2}}$$

For fast order-of-magnitude estimates, we need some g_*. For $T \sim$ MeV:

$$g_* \approx \underset{\gamma}{2} + \frac{7}{8}(\underset{e^\pm}{4} + \underset{\nu\text{'s}}{3 \times 2}) = 2 + \frac{70}{8} = 10.75 .$$

For $T \gg 100$ GeV:

$$g_* \approx \underset{\gamma}{2} + \underset{g}{16} + \underset{W^\pm, Z^0}{3\times 3} + \underset{h}{1} + \frac{7}{8}(\underset{e,\mu,\tau}{3\times 4} + \underset{\nu\text{'s}}{3\times 2} + \underset{\text{quarks}}{3 \times 8 \times N_c}) = 106.75 .$$

Setting $g_* \sim 10$ and being careful about units (we use $\hbar = c = k_B = 1$) produces finally

$$\frac{t}{s} \sim \left(\frac{\text{MeV}}{T}\right)^2 .$$

Big picture (1eV = 11600K)

T	what happens	t
10^{15} GeV	inflation ?	$\sim 10^{-35}$ s
PeV		
TeV		
	electroweak phase transition	10^{-10} s
GeV		
	QCD phase transition	10^{-5} s
MeV	neutrino decoupling	1 s
	e^+e^- annihilation	
	nucleosynthesis (BBN)	10^2 s
keV		
eV (plasma ↑)	photon decoupling (CMB)	300000 y
(gas ↓)	21 cm line	
	galaxies (LSS), reionisation, supernovae, heavy elements, ...	
10^{-3} eV	„life"	13×10^9 y

density perturbations