8.2. Hard Thermal Loop Effective Theory

We saw earlier on (Sec. 5.4; pp. 74–77) that in the static limit, the Matsubara zero-modes are the only ones which are IR sensitive. By loop corrections, a thermal mass is however generated for A_0^a. The mass appears for instance in the dimensionally reduced low-energy effective field theory for QCD (sec 6.3; pp. 90–92), which can be used to compute reliably all static observables.

Question: How do these concepts generalise to real-time observables?

Answer: It gets much more complicated!

The basic reason is that so far $\frac{1}{\omega_n^2+E^2}$ was largest for $\omega_n = 0$. Now, however, we are looking at

$$\frac{1}{\omega_n^2+E^2} = \frac{1}{2E}\left(\frac{1}{i\omega_n+E} + \frac{1}{-i\omega_n+E}\right) \xrightarrow{i\omega_n \to \omega + i\epsilon} \frac{1}{2E}\left(\frac{1}{\omega+E+i\epsilon} + \frac{1}{-\omega+E-i\epsilon}\right).$$

Thus, after analytic continuation, the propagator is large at $\omega \approx \pm E$, i.e. close to on-shell. But this is a genuinely four-dimensional concept.

Nevertheless, a low-energy effective theory can still be constructed.
[R.D. Pisarski, Phys. Rev. Lett. 63 (1989) 1129; E. Braaten and R.D. Pisarski, Phys. Rev D 45 (1992) 1827]

* Consider "soft" external frequencies (p_0) and momenta ($\vec{p}$), $|p_0|, |\vec{p}| \lesssim gT$.
* Inside the loops, sum over all frequencies ω_n.
* And integrate over "hard" spatial momenta, $|\vec{p}| \gtrsim T$.

$\Rightarrow$ such loops are called "hard thermal loops" (HTL).

Let us see what it gives for QCD! We can directly use the results on pp. 74-77, with the following in mind:

* Since external momenta are "soft", we can put $P \to 0$ in the numerators, just as we did there $(|p_0|, |\vec{p}| \lesssim gT)$.
* We should not put $P \to 0$ in the denominators, however, as we just saw that the denominators can become singular after analytic continuation!

Expressed as a correction to the inverse propagator (P^2) we get, in the Feynman gauge $\xi = 1$:

$$g^2 N_c \sum\!\!\!\!\!\!\!\int \frac{d}{R^2}\,\delta_{\mu\nu}$$

$$g^2 N_c \sum\!\!\!\!\!\!\!\int_R \left[-\frac{1}{R^2}\delta_{\mu\nu} - (2d-1)\frac{R_\mu R_\nu}{R^2 (R+P)^2} \right]$$

$$g^2 N_c \sum\!\!\!\!\!\!\!\int_R \left[\frac{R_\mu R_\nu}{R^2 (R+P)^2} \right]$$

$$\Rightarrow \quad \text{sum} = \underset{d=3}{2} g^2 N_c \sum\!\!\!\!\!\!\!\int_R \left[\frac{\delta_{\mu\nu}}{R^2} - \frac{2 R_\mu R_\nu}{R^2 (R+P)^2} \right]$$

Next step: carry out $T\sum_{r_n}$. It's familiar by now: $\frac{1}{2}\int_{-\infty-i\epsilon}^{\infty-i\epsilon} \frac{d\omega}{2\pi}\left[f(\omega)+f(-\omega)\right]\left[1+2n_b(i\omega)\right]$.

Denote $E_1^2 \equiv \vec{r}^{\,2}$, $E_2^2 \equiv (\vec{r}+\vec{p})^2$.

<u>First part</u>: (actually we know the answer, but let us redo it this way)

$$\sum\!\!\!\!\!\!\!\int_R \frac{1}{R^2} = T\sum_{r_n}\int \frac{d^3r}{(2\pi)^3}\frac{1}{(r_n^2 + E_1^2)}$$

$$= \int \frac{d^3r}{(2\pi)^3}\left\{ \frac{1}{2\pi}\cdot(-2\pi i)\,\frac{1}{-2iE_1}\cdot\left(1+2n_b(E_1)\right) \right\}$$

$$= \int \frac{d^3r}{(2\pi)^3}\left\{ \frac{1}{2E_1} + \frac{n_b(E_1)}{E_1} \right\}\Bigg|_{E_1 = |\vec{r}|}.$$

Second part : From 110-111 we know that

$$T \sum_{r_n} \frac{1}{[r_n^2 + E_1^2][(r_n+p_n)^2 + E_2^2]} \, g(r_n, \bar{r})$$

$$= \frac{1}{(2E_1)(2E_2)} \Bigg\{ \frac{1}{-ip_n - E_1 + E_2} \Big[\left(\tfrac{1}{2} + n_b(E_1)\right) g(-iE_1, \bar{r}) - \left(\tfrac{1}{2} + n_b(E_2)\right) g(-p_n - iE_2, \bar{r}) \Big]$$

$$+ \frac{1}{ip_n + E_1 + E_2} \Big[\left(\tfrac{1}{2} + n_b(E_1)\right) g(-iE_1, \bar{r}) + \left(\tfrac{1}{2} + n_b(E_2)\right) g(p_n - iE_2, \bar{r}) \Big]$$

$$+ \frac{1}{-ip_n + E_1 + E_2} \Big[\left(\tfrac{1}{2} + n_b(E_1)\right) g(-iE_1, \bar{r}) + \left(\tfrac{1}{2} + n_b(E_2)\right) g(-p_n - iE_2, \bar{r}) \Big]$$

$$+ \frac{1}{ip_n - E_1 + E_2} \Big[\left(\tfrac{1}{2} + n_b(E_1)\right) g(-iE_1, \bar{r}) - \left(\tfrac{1}{2} + n_b(E_2)\right) g(p_n - iE_2, \bar{r}) \Big] \Bigg\}$$

Here the object $g(r_n, \bar{r}) \equiv -2 R_\mu R_\nu$ is a polynomial in both arguments. Moreover mixed indices cancel, as the integrand would be odd in $\bar{r} \to -\bar{r}$.

Note:
- $E_1 = |\bar{r}|$; $|\bar{r}| = \text{hard} \sim T$
- $E_2 = |\bar{r} + \bar{p}| = (\bar{r}^2 + 2\bar{r}\cdot\bar{p} + \bar{p}^2)^{\frac{1}{2}} = |\bar{r}| + \frac{\bar{r}\cdot\bar{p}}{\bar{r}} + O(\bar{p})^2$; $|\bar{p}| = \text{soft} \leq gT$.

  Denote $\frac{r_i}{|\bar{r}|} \equiv v_i \Rightarrow E_2 = |\bar{r}| + \bar{v}\cdot\bar{p} + O(\bar{p})^2$.
- After analytic continuation, $ip_n \to p_0 + i\epsilon \equiv \text{soft} \leq gT$.
- $d^3\bar{r} \sim T^3$
- $\frac{1}{(2E_1)(2E_2)} \sim \frac{1}{T^2}\left(1 + O\left(\frac{|\bar{p}|}{T}\right)\right)$
- For simplicity, let us consider spatial components only.

  Then $g = -2\bar{r}_i\bar{r}_j \sim T^2$.
- For $\frac{1}{\pm ip_n + E_1 + E_2} \sim \frac{1}{T}$, we get the leading term $\sim T^2$ by taking everywhere the zeroth order in $O(\bar{p})$.
- For $\frac{1}{\pm ip_n - E_1 + E_2} \sim \frac{1}{gT}$, we have to expand everything to first order in $O(\bar{p})$!

Expansions:

$$* \quad \frac{1}{2E_2} = \frac{1}{2|\bar{r}|}\left(1 - \frac{\bar{v}\cdot\bar{p}}{|\bar{r}|}\right) + \dots$$

$$* \quad \frac{1}{\mp ip_n - E_1 + E_2} = \frac{1}{\mp ip_n + \bar{v}\cdot\bar{p}} + \dots$$

$$* \quad n_b(E_2) = n_b(|\bar{r}|) + \bar{v}\cdot\bar{p}\,\frac{\partial n_b(|\bar{r}|)}{\partial|\bar{r}|} + \dots \qquad ; \text{ denote } r = |\bar{r}|$$

$$\Rightarrow \quad -2r_ir_j \int\frac{d^3r}{(2\pi)^3}\frac{1}{(2r)^2}\Big(1 - \underbrace{\cancel{\tfrac{\bar{v}\cdot\bar{p}}{r}}}_{\mathcal{O}(\bar{p})}\Big)\Bigg\{\frac{1}{-ip_n + \bar{v}\cdot\bar{p}}\left[-\frac{\partial n_b}{\partial r}\,\bar{v}\cdot\bar{p}\right]$$

$$+ \frac{1}{2r}\left[1 + 2n_b(r)\right]$$

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$$+ \frac{1}{ip_n + \bar{v}\cdot\bar{p}}\left[-\frac{\partial n_b}{\partial r}\,\bar{v}\cdot\bar{p}\right]\Bigg\}$$

$$= \int\frac{d^3r}{(2\pi)^3}\cdot\left(\frac{-r_ir_j}{r^2}\right)\left\{\frac{\bar{v}\cdot\bar{p}}{ip_n - \bar{v}\cdot\bar{p}}\frac{\partial n_b}{\partial r} + \frac{1}{r}\left[\frac{1}{2} + n_b(r)\right]\right\}$$

Now write $\bar{v}\cdot\bar{p} = -ip_n + \bar{v}\cdot\bar{p} + ip_n$, insert $ip_n \to p_0 + i\varepsilon$, and combine with the other term. Leaving out a vanishing $i\varepsilon$ from the numerator, we get:

$$-2g^2N_c\int\frac{d^3r}{(2\pi)^3}\frac{\partial n_b}{\partial r}\frac{p_0 v_i v_j}{p_0 + i\varepsilon - \bar{v}\cdot\bar{p}}$$

$$\langle v_iv_j\rangle = \tfrac{1}{3}\delta_{ij}$$

$$+ 2g^2N_c\int\frac{d^3r}{(2\pi)^3}\left\{\frac{1}{2r} + \frac{n_b(r)}{r} - \frac{1}{3}\left[-\frac{\partial n_b}{\partial r} + \frac{1}{2r} + \frac{n_b(r)}{r}\right]\right\}\cdot\delta_{ij} \qquad (*)$$

from p. 116

The integrals: $* \quad d^3r \equiv \underbrace{r^2dr\,d\Omega}_{\text{angular part}} \equiv 4\pi r^2dr\,\frac{d\Omega}{4\pi}$

$* \quad \int d^3r\cdot\frac{1}{r} = 0$ (in dim. reg.; but these T-independent terms could be seen to cancel with a careful analysis at $T=0$.)

$$* \quad \int\frac{d^3r}{(2\pi)^3}\frac{\partial n_b}{\partial r} = \frac{1}{2\pi^2}\int_0^\infty dr\,r^2\frac{dn_b}{dr} = -2\cdot\frac{1}{2\pi^2}\int_0^\infty dr\,r^2\frac{n_b}{r}$$

$$= -\frac{1}{\pi^2}\int_0^\infty dr\,r\,\frac{1}{e^{\frac{r}{T}} - 1} = -\frac{T^2}{6}.$$

$*$ the latter term in $(*)$: $\int\frac{d^3r}{(2\pi)^3}\left[\frac{2}{3}\frac{n_b}{r} + \frac{1}{3}\frac{\partial n_b}{\partial r}\right] = 0$!

Summary:

* For the spatial components, we obtain the correction

$$\Pi_{ij} = g^2T^2\frac{N_c}{3}\int\frac{d\Omega}{4\pi}\cdot\frac{p_0 v_i v_j}{p_0 - v_i p_i + i\epsilon} \qquad \text{to } (-p_0^2+\bar{p}^2)\,\delta_{ij} .$$

This is the famous HTL self-energy. A very non-trivial dependence on $p_0, |\bar{p}|$!

* With fermions: $g^2T^2\frac{N_c}{3} \to m_D^2 = g^2T^2\left(\frac{N_c}{3}+\frac{N_f}{6}\right)$.

* Limits : • static $\Rightarrow p_0 = 0 \Rightarrow 0 \Rightarrow$ no magnetic screening at $O(g^2T^2)$!
  • homogeneous $\Rightarrow p_i = 0 \Rightarrow \delta_{ij}\left(-p_0^2+\frac{1}{3}m_D^2\right) \Rightarrow$ "electric plasma oscillations". $[\partial_0 A_i \sim E_i]$

* Generalization : (add zero components ; make gauge-invariant)

$$\mathcal{L}_M = -\frac{1}{2}\mathrm{Tr}\left[F_{\mu\nu}F^{\mu\nu}\right] + \frac{1}{2}m_D^2\int\frac{d\Omega}{4\pi}\,\mathrm{Tr}\left[\left(\frac{1}{v\cdot D^{adj}}v^\alpha F_{\alpha\mu}\right)\left(\frac{1}{v\cdot D^{adj}}v^\beta F_\beta{}^\mu\right)\right],$$

written as it would appear with $ip_n \to p_0$

$$v \equiv (1, \bar{v}).$$

what does this mean precisely?

The propagators and vertices derived from here are to be used as an effective low-energy theory.

* Strength: more general than the DR theory, as allows to address both static and time-dependent observables.

* Weakness: it's non-local !?? $\Rightarrow$ the natural low-energy d.o.f.'s have not been identified (i.e., some have been integrated out)

  $\Rightarrow$ What kind of higher order operators are there ?

  $\Rightarrow$ accuracy ?? Correction are $O(g)$ or smaller ? How to include $O(g), O(g^2), \ldots$?

  $\Rightarrow$ non-renormalisable.

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