7.9. Effective potential; Bose-Einstein condensation

To treat the infrared dynamics properly, let us introduce for a moment a finite volume $V = L_1 L_2 L_3$. Then spatial momenta get discretised, $\bar{p} = 2\pi\left(\frac{n_1}{L_1}, \frac{n_2}{L_2}, \frac{n_3}{L_3}\right)$.

The mode with $\bar{p} = 0$, $\omega_n^b = 0$ $\equiv$ condensate $\equiv \bar{\phi}$

We can write:
$$Z(T,\mu) = \int_{-\infty}^{\infty} d\bar{\phi} \int_{\text{periodic}, \, p \neq 0} \mathcal{D}\phi' \; e^{-S_E[\phi = \bar{\phi} + \phi']}$$
$$\equiv \int_{-\infty}^{\infty} d\bar{\phi} \; \exp\left[-\frac{V}{T} \cdot V_{eff}(\bar{\phi})\right]$$

Here $V_{eff} \equiv$ (constrained) effective potential.

For our action, setting $\lambda = 0$ for simplicity, and noting that $\int_0^\beta d\tau \int d^d x \, \phi' = 0$:
$$S_E[\phi = \bar{\phi} + \phi'] = \frac{V}{T}(m^2 - \mu^2)\, \bar{\phi}^* \bar{\phi} + \underbrace{\sum\!\!\!\!\!\!\int_{P \neq 0} \tilde{\phi}^*(P) \tilde{\phi}(P) \left[(\omega_n + i\mu)^2 + \bar{p}^2 + m^2\right]}_{\text{constant}} .$$

$$\Rightarrow V_{eff}(\bar{\phi}) = (m^2 - \mu^2)\bar{\phi}^* \bar{\phi} + \text{constant}$$

If L, m are finite and $\mu \le m$, then for $T \to 0$ the constant term can be neglected.

Consider also charge density:
$$Z = \mathrm{Tr}\, e^{-\beta \hat{H} + \beta\mu \hat{Q}} \quad \Rightarrow \quad \frac{\partial \ln Z}{\partial \mu} = \langle \beta \hat{Q} \rangle$$
$$\Rightarrow \rho = \frac{\langle \hat{Q} \rangle}{V} = \frac{T}{V}\frac{\partial \ln Z}{\partial \mu} = \frac{\int_{-\infty}^{\infty} d\bar{\phi} \; 2\mu \bar{\phi}^* \bar{\phi} \; \exp\left[-\frac{V}{T} V_{eff}(\bar{\phi})\right]}{\int_{-\infty}^{\infty} d\bar{\phi} \; \exp\left[-\frac{V}{T} V_{eff}(\bar{\phi})\right]}$$

<u>Now:</u>

- $*$ $\mu > m$ $\Rightarrow$ integrals not defined $\Rightarrow$ unphysical situation.
- $*$ $\mu < m$ $\Rightarrow$ $\rho \to 0$ for $T \to 0$ ($\partial_\mu \ln Z$ is finite & regular)
- $*$ $\mu = m$ $\Rightarrow$ borderline case. Requires a careful analysis, but in the end the limit can be taken by keeping ρ fixed. We get

$$\rho = 2m \, \underbrace{\langle \bar{\phi}^* \bar{\phi} \rangle}_{\text{"condensate", BEC}} > 0 .$$

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7.3 Dirac fermions with a chemical potential

The quark Lagrangian of QCD,

$$\mathcal{L} = \bar{\Psi}_{jA}\left[i\,\not{D}_{AB}\,\delta_{jk} - M_{jk}\delta_{AB}\right]\Psi_{kB} \quad , \quad \not{D}_{AB} = \gamma^\mu\left(\partial_\mu \delta_{AB} - ig A^a_\mu T^a_{AB}\right)$$

has also a global symmetry: $\Psi_{kB} \to e^{i\alpha}\Psi_{kB}$, $\bar{\Psi}_{jA} \to e^{-i\alpha}\bar{\Psi}_{jA}$.

The Noether current:

$$j_\mu = \frac{\partial \mathcal{L}}{\partial(\partial^\mu \Psi_{kB})}\frac{\delta \Psi_{kB}}{\delta \alpha} = \bar{\Psi}_{kB}\, i\gamma_\mu \; i\Psi_{kB} = -\bar{\Psi}_{kB}\gamma_\mu \Psi_{kB} .$$

Again $[\hat{H}, \hat{Q}] = 0$ and the charge can be treated as a part of the Hamiltonian:

$$\hat{H} - \mu\hat{Q} \to \mathcal{H} - \mu Q \quad , \quad \mu Q = -\mu\int d^dx\; \bar{\Psi}_{kB}\gamma_0\Psi_{kB} .$$

$$\Rightarrow \quad Z(T,\mu) = C \underset{\text{antiperiodic}}{\int} \mathcal{D}\bar{\Psi}\mathcal{D}\Psi \exp\left\{-\int_0^\beta d\tau\int d^dx \left[\bar{\Psi}\left(\gamma_\mu D_\mu + \gamma_0 \mu + M\right)\Psi\right]\right\} .$$

The quadratic part in momentum space:

$$\underset{\omega_n f}{\sum\!\!\!\!\!\!\!\int} \bar{\Psi}(\omega_n,\bar{p})\left[i\gamma_0\omega_n + i\gamma_i p_i + \gamma_0\mu + M\right]\Psi(\omega_n,\bar{p})$$

$\Rightarrow$ Just like for bosons, simply shift $\underline{\underline{\omega_n \to \omega_n - i\mu\,!}}$

The propagator etc remain the same.

whether + or − has no actual significance.

What is the free energy density for a single Dirac fermion of mass m?

* Recall first again the trivial factors:

Bosons: $\int \mathcal{D}\phi \, e^{-\phi^* [p^2+m^2]\phi} \propto \frac{1}{p^2+m^2} = e^{-\ln(p^2+m^2)}$

Fermions: $\int \mathcal{D}\bar\psi \mathcal{D}\psi \, e^{-\bar\psi[i\not p + m]\psi} \propto \mathrm{Det}[i\not p + m] = (p^2+m^2)^2 = e^{2\ln(p^2+m^2)}$

$\Rightarrow$ A Dirac fermion has an overall factor -2 with respect to a complex scalar field.

* Moreover, since μ appears in exactly the same ways, we can apply the result $S_f(T) = 2S_b(\frac{T}{2}) - S_b(T)$:

$$f(T,\mu) = -2\int\frac{d^dp}{(2\pi)^d}\Big\{E + T\Big[\ln\big(1-e^{-2\beta(E-\mu)}\big) + \ln\big(1-e^{-2\beta(E+\mu)}\big) - \ln\big(1-e^{-\beta(E-\mu)}\big) - \ln\big(1-e^{-\beta(E+\mu)}\big)\Big]\Big\}$$

$$= -2\int\frac{d^dp}{(2\pi)^d}\Big\{E + T\Big[\ln\big(1+e^{-\frac{E-\mu}{T}}\big) + \ln\big(1+e^{-\frac{E+\mu}{T}}\big)\Big]\Big\}, \quad E=\sqrt{\vec p^{\,2}+m^2}.$$

Note that this integral is well-defined for any μ

$\Rightarrow$ no IR-problems / condensation, like for bosons!

[Unless interactions bind fermions to a bosonic bound state.]

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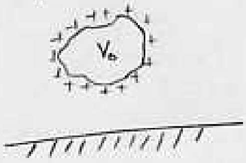
What about gauge symmetries?

We know that electric charge is also conserved; can we associate μ with it?

Physics: If a conductor is charged, then the charge resides on the surface, because charged particles interact with a repulsive long-range force. That is, the homogeneous "bulk" of the conductor is neutral. It has, however, an electric potential with respect to the ground:



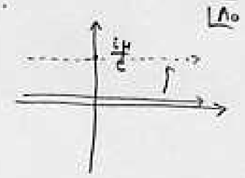
Formally: For a gauged $U(1)$ (QED), we get

$$Z(T,\mu) = C_1 \int \mathcal{D}A_\mu \mathcal{D}\bar{\psi}\mathcal{D}\psi \exp\left\{ -\int_0^\beta d\tau \int d^dx \left[\frac{1}{4}F_{\mu\nu}^2 + \bar{\psi}\left(\gamma_0(\partial_0 - ieA_0 - \mu) + \gamma_i D_i + m\right)\psi \right]\right\}.$$

The claim is that in a homogeneous system, this result does not depend on μ, so that $\frac{Q}{V} \propto \frac{\partial f}{\partial \mu} = 0$.

Why?

(a) The path integral involves an integral over the zero momentum-mode of A_0, $\bar{A}_0$. It can formally be shifted in the complex plane such that μ is eaten away: $\bar{A}_0^{(\text{new})} = \bar{A}_0 + \frac{i\mu}{e}$



(b) Like on p. 99, we could start by computing the effective potential for $\bar{A}_0$. It has a minimum at a non-trivial value (which is actually imaginary; this corresponds to a real Minkowskian $\bar{A}_{0,M} \sim V_0$). The value of the potential at the minimum cancels against the μ-dependence of the free fermion determinant (see next page), such that the total $f(T,\mu)$ is μ-independent.

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