6.3 The dimensionally reduced effective theory for hot QCD

Let us apply the program on p. 89 to high-temperature QCD!

(1) The light degrees of freedom are as discussed on p. 86: bosonic Matsubara zero-modes. Since they do not depend on τ, they live in three dimensions only: therefore the construction of the effective theory is called "dimensional reduction".

[Ginsparg, Nucl. Phys. B 170 (1980) 388;
Appelquist - Pisarski, Phys. Rev. D 23 (1981) 2305.]

(2) What are the symmetries?

* Since the τ-direction of the original theory is different from the space directions, the theory needs not be fully covariant in space-time transformations. It only needs to be so in spatial rotations and translations.

* The original theory has the discrete symmetries CPT. For instance, it is invariant in $A_0^a \to -A_0^a$!

  [Proof: exercise.]

* Consider then gauge symmetries.

$$\begin{cases} A_\mu \to A'_\mu = U A_\mu U^{-1} + \frac{i}{g} U \partial_\mu U^{-1} \quad , \quad A_\mu = A_\mu^a T^a \\ A_\mu^a \to A_\mu^{a\prime} = A_\mu^a + \partial_\mu \theta^a + g f^{abc} A_\mu^b \theta^c + O(\theta^2) \end{cases}$$

  Since we restrict to static (τ-independent) fields, U (or θ^a) should not depend on τ either! Thus, the effective theory should be invariant under

$$\begin{cases} A_i \to A'_i = U A_i U^{-1} + \frac{i}{g} U \partial_i U^{-1} & \text{(normal gauge transformation)} \\ A_0 \to A'_0 = U A_0 U^{-1} & \text{("scalar field in the adjoint representation")} \end{cases}$$

We are now ready to write down the effective theory:

(a) Tree-level (leaving out gauge fixing and ghosts for simplicity):

$$\mathcal{L}_E = \frac{1}{4} F^a_{\mu\nu} F^a_{\mu\nu}$$

$$\rightarrow \frac{1}{4} F^a_{ij} F^a_{ij} + \frac{1}{2} F^a_{i0} F^a_{i0}$$

Here $F^a_{i0} = \partial_i A^a_0 - \underbrace{\cancel{\partial_0 A^a_i}}_{0 \text{ (static!)}} + g f^{abc} A^b_i A^c_0$

$$= \left(\partial_i \delta^{ac} + g f^{abc} A^b_i \right) A^c_0 \equiv D^{ac}_i A^c_0$$

covariant derivative in <u>adjoint</u> representation!

Note that: $T^a F^a_{i0} = \partial_i A_0 + g f^{abc} T^a A^b_i A^c_0$

$$= \partial_i A_0 - ig \left[A^b_i, A^c_0 \right] \equiv \left[D_i, A_0 \right]$$

$D_i = \partial_i - igA_i$ = covariant derivative in <u>fundamental</u> representation!

$$\Rightarrow \mathcal{L}_E = \frac{1}{4} F^a_{ij} F^a_{ij} + \mathrm{Tr} \left[D_i, A_0 \right] \left[D_i, A_0 \right]$$

(b) Other gauge-invariant operators that could appear:

dim = 2: $\mathrm{Tr}\left[A_0^2 \right]$

dim = 4: $\mathrm{Tr}\left[A_0^4 \right]$, $\left(\mathrm{Tr}\left[A_0^2 \right] \right)^2$

dim = 6: $\mathrm{Tr}\left\{ \left[D_i, F_{ij} \right]^2 \right\}$, etc. There are very many; classified by S. Chapman, Phys. Rev. D 50 (1994) 5308.

(c) The action:

$$S_E = \frac{1}{T} \int d^3x \left\{ \frac{1}{4} F^a_{ij} F^a_{ij} + \mathrm{Tr}\left[D_i, A_0 \right]\left[D_i, A_0 \right] + m^2 \, \mathrm{Tr}\left[A_0^2 \right] + \lambda^{(1)} \left(\mathrm{Tr}\left[A_0^2 \right] \right)^2 + \lambda^{(2)} \, \mathrm{Tr}\left[A_0^4 \right] + \ldots \right\}.$$

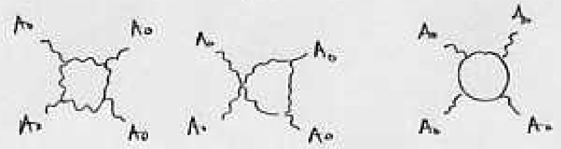
from $\int_0^\beta d\tau$

3. Matching for the coefficients:

* On p. 74-77, we computed the effective thermal mass for A_0^a, at a vanishing momentum. This is precisely m^2!

$$\Rightarrow \quad m^2 = g^2T^2\left(\frac{N_c}{3} + \frac{N_f}{6}\right) + \mathcal{O}(g^4T^2)$$

* $\lambda^{(i)}$ arise from graphs like



etc.

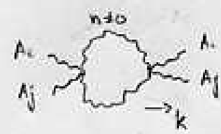
They are thus parametrically $\sim g^4$.

The actual values [N.P. Landsman, Nucl. Phys. B 322 (1989) 498]:

$$\lambda^{(1)} = \frac{g^4}{4\pi^2}, \quad \lambda^{(2)} = \frac{g^4}{12\pi^2}(N_c - N_f)$$

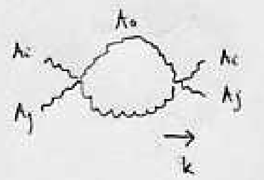
4. Estimating the error from higher order terms:

Compute the operators neglected, e.g.



$$\sim \frac{g^4}{T^2}(\partial_i F_{ij}^a)^2 ,$$

to dynamical effects within the effective theory, e.g.



$$\sim g^4 (\partial_i F_{ij}^a)^2 \, T \int \frac{d^3p}{(2\pi)^3} \frac{1}{(p^2+m^2)^3}$$

$$\sim \frac{g^4 T}{m^3}(\partial_i F_{ij}^a)^2$$

$\Rightarrow$ The error made is $\sim \dfrac{g^4}{T^2} \cdot \dfrac{m^3}{g^4 T} \sim \left(\dfrac{m}{T}\right)^3 \sim g^3$.

$\Rightarrow$ This generic estimate is not always valid, however.
There could be, say, a situation where the contribution within the effective theory vanishes due to some accidental symmetry that has been introduced, and then the error is large.

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