Summary: Basic relations for fermions

$$\hat{O}(t) \equiv e^{i\hat{H}t}\,\hat{O}(0)\,e^{-i\hat{H}t}$$
$$\hat{O}(\tau) \equiv e^{\hat{H}\tau}\,\hat{O}(0)\,e^{-\hat{H}\tau}$$

Let: $\hat{A}(t) \equiv \hat{\psi}_\alpha(t,\bar{x})$; $\hat{B}(0) \equiv \hat{\bar{\psi}}_\beta(0,\bar{y})$.

Thermal ensemble : $\hat{\rho} = \frac{1}{Z} e^{-\beta(\hat{H}-\mu\hat{Q})}$.

Sets of correlators:

(A)

$$S^>(p_0) \equiv \int_{-\infty}^{\infty} dt\, e^{ip_0 t} \left\langle \hat{A}(t)\hat{B}(0) \right\rangle$$

$$S^<(p_0) \equiv \int_{-\infty}^{\infty} dt\, e^{ip_0 t} \left\langle -\hat{B}(0)\hat{A}(t) \right\rangle$$

$$\rho(p_0) \equiv \int_{-\infty}^{\infty} dt\, e^{ip_0 t} \left\langle \tfrac{1}{2}\left\{ \hat{A}(t), \hat{B}(0) \right\} \right\rangle$$

(C)

$$S_T(p_0) \equiv \int_{-\infty}^{\infty} dt\, e^{ip_0 t} \left\langle \hat{A}(t)\hat{B}(0)\,\theta(t) - \hat{B}(0)\hat{A}(t)\,\theta(-t) \right\rangle ;$$

$$\theta(t) = i\int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} \cdot \frac{e^{-i\omega' t}}{\omega' + i\epsilon}$$

(D)

$$S_E'(\omega_n) \equiv \int_0^\beta d\tau\, e^{(i\omega_n+\mu)\tau} \left\langle \hat{A}(\tau)\hat{B}(0) \right\rangle$$

Note: $e^{\mu\tau}\langle \hat{A}(\tau)\hat{B}(0)\rangle$ is antiperiodic over β provided that $\{\hat{A}(0),\hat{B}(0)\} = 0$ & $[\hat{A},\hat{Q}] = \hat{A}$.

Relations:

* Within (A): • note that $\langle n|\hat{\mathcal{U}}|m\rangle\langle m|\hat{B}|n\rangle = \langle m|\hat{B}|n\rangle\langle n|\hat{\mathcal{U}}|m\rangle$,
since one of the states is necessarily Grassmannian.

• using
$$\hat{\mathcal{U}} = \hat{\Psi} = \int \frac{d^3\bar{p}}{\sqrt{(2\pi)^3 2E_p}} \sum_s \left[\hat{a}_{\bar{p}}^{(s)} u(\bar{p},s) e^{-ip\cdot x} + \hat{b}_{\bar{p}}^{+(s)} v(\bar{p},s) e^{+ip\cdot x} \right]$$
$$\hat{Q} = \int d^3\bar{q} \sum_t \left[\hat{a}_{\bar{q}}^{+(t)} \hat{a}_{\bar{q}}^{(t)} - \hat{b}_{\bar{q}}^{+(t)} \hat{b}_{\bar{q}}^{(t)} \right]$$
$$\{ \hat{a}_{\bar{p}}^{(s)}, \hat{a}_{\bar{q}}^{+(t)} \} = \{ \hat{b}_{\bar{p}}^{(s)}, \hat{b}_{\bar{q}}^{+(t)} \} = \delta_{st}\, \delta^{(3)}(\bar{p}-\bar{q})$$
find $\hat{Q}\hat{\mathcal{U}} = \hat{\mathcal{U}}(\hat{Q}+1)$!

Then:
$$S^> = -e^{\beta(p_0-\mu)} S^<$$
$$\Rightarrow \quad S^>(p_0) = 2\left[1-n_F(p_0-\mu)\right]\rho(p_0) \qquad ; \quad n_F(x) = \frac{1}{e^{\beta x}+1}$$
$$S^<(p_0) = 2\left[-n_F(p_0-\mu)\right]\rho(p_0)$$

* From (A) to (C):
$$S_T(p_0) = \int_{-\infty}^{\infty} \frac{d\omega}{\pi} \frac{i\rho(\omega)}{p_0-\omega+i\varepsilon} - 2\rho(p_0)\, n_F(p_0-\mu)$$

* From (D) to (A):
$$\rho(\omega) = \mathrm{Im}\, S_E'\left(i\omega_n \to \omega-\mu+i\varepsilon\right).$$

* Example:
$$S_E'(\omega_n) = \frac{A\cdot i(\omega_n-i\mu)+B}{(\omega_n-i\mu)^2+E^2}$$
$$\rho(\omega) = \frac{\pi}{2E}\left\{\delta(\omega-E)-\delta(\omega+E)\right\}\left\{A\omega+B\right\}$$
Note: independent of T & μ !
$$S_T(p) = \left\{Ap_0+B\right\}\left\{\frac{i}{p_0^2-E^2+i\varepsilon} - 2\pi\,\delta(p_0^2-E^2)\, n_F\left[|p_0|-\mathrm{sign}(p_0)\mu\right]\right\}.$$

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