Exercise 12 :

Starting from the expression for $\mathcal{L}_H$ on p. 119, take the quadratic part in momentum space and:

(a) verify that in the static limit $(p_0 \to 0)$, the zero-components A_0^a get the familiar Debye mass.

(b) verify that the spatial components have the self-energy correction Π_{ij} on p. 119.

Solution :

- $\mathrm{Tr}\left[F_{\mu\nu}F^{\mu\nu}\right] = \frac{1}{2} F_{ij}^a F_{ij}^a - F_{0i}^a F_{0i}^a$

- $\mathrm{Tr}\left[\left(\frac{1}{v\cdot\partial} v^\alpha F_{\alpha\mu}\right)\left(\frac{1}{v\cdot\partial} v^\beta F_\beta{}^\mu\right)\right]$

$$= \frac{1}{2}\left(\frac{1}{v\cdot\partial} v^\alpha F_{\alpha 0}^a\right)\left(\frac{1}{v\cdot\partial} v^\beta F_{\beta 0}^a\right) - \frac{1}{2}\left(\frac{1}{v\cdot\partial} v^\alpha F_{\alpha i}^a\right)\left(\frac{1}{v\cdot\partial} v^\beta F_{\beta i}^a\right)$$

- $F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a = \left(\partial_\mu \delta_{\nu\alpha} - \partial_\nu \delta_{\mu\alpha}\right) A_\alpha^a$

Thus, in momentum space:

$$-\frac{1}{2}\mathrm{Tr}\left[F_{\mu\nu}F^{\mu\nu}\right] \to -\frac{1}{4}\left(-ip_i\delta_{j\alpha} + ip_j\delta_{i\alpha}\right)\left(ip_i\delta_{j\beta} - ip_j\delta_{i\beta}\right)A_\alpha^a(-p)A_\beta^a(p)$$
$$+\frac{1}{2}\left(-ip_0\delta_{i\alpha} + ip_i\delta_{0\alpha}\right)\left(ip_0\delta_{i\beta} - ip_i\delta_{0\beta}\right)A_\alpha^a(-p)A_\beta^a(p)$$

$$= -\frac{1}{2}\left(\bar{p}^2\delta_{ij} - p_ip_j\right)A_i^a(-p)A_j^a(p)$$
$$+\frac{1}{2}p_0^2 A_i^a(-p)A_i^a(p) + \frac{1}{2}\bar{p}^2 A_0^a(-p)A_0^a(p)$$
$$-\frac{1}{2}p_ip_0\left[A_0^a(-p)A_i^a(p) + A_i^a(-p)A_0^a(p)\right]$$

$\Rightarrow$ (a) In the static limit, $\partial_0 = 0$, $p_0 = 0$

$$\Rightarrow \quad v^\alpha F_{\alpha 0}^a = v\cdot\partial\, A_0^a$$

$$v^\alpha F_{\alpha i}^a = \bar{v}\cdot\bar{\partial}\, A_i^a - \partial_i v^\alpha A_\alpha^a$$

$$\frac{1}{v\cdot\partial} = \frac{1}{\bar{v}\cdot\bar{\partial}}$$

$$\Rightarrow \quad \frac{1}{4} m_D^2 \int \frac{d\Omega}{4\pi} \cdot \left\{ A_0^a(-P) A_0^a(P) - \left[A_i^a(-P) + \frac{i p_i\, v^\alpha A_\alpha^a(-P)}{-i\bar{v}\cdot\bar{p}} \right]\left[A_i^a(P) - \frac{i p_i\, v^\beta A_\beta^a(P)}{i\bar{v}\cdot\bar{p}} \right] \right\}$$

Pick out zero-component part :

$$\frac{1}{4} m_D^2 \int \frac{d\Omega}{4\pi} \cdot A_0^a(-P) A_0^a(P) \cdot \left(1 - \frac{\bar{p}^2}{(\bar{v}\cdot\bar{p})^2} \right)$$

This, however, is hopelessly divergent! Have to take the limit $p_0 \to 0$ more carefully :

$$\lim_{p_0\to 0} \int \frac{d\Omega}{4\pi} \frac{\bar{p}^2}{(p_0 + i\varepsilon - \bar{v}\cdot\bar{p})^2} = \lim_{\delta\to 0} \frac{1}{2} \int_{-1}^{+1} d(\cos\theta) \frac{1}{(\delta + i\varepsilon' - \cos\theta)^2} \quad , \; \cos\theta = \frac{\bar{p}\cdot\bar{v}}{|\bar{p}|} \, , \; \delta = \frac{p_0}{|\bar{p}|}$$

$$\varepsilon' = \frac{\varepsilon}{|\bar{p}|}$$

$$= \lim_{\delta\to 0} \frac{1}{2} \int_{-1}^{+1} dz \frac{1}{(\delta + i\varepsilon' - z)^2}$$

$$= \lim_{\delta\to 0} \frac{1}{2} \left. \frac{1}{\delta + i\varepsilon' - z} \right|_{-1}^{+1}$$

$$= \lim_{\delta\to 0} \frac{1}{2} \left\{ \frac{1}{\delta - 1 + i\varepsilon'} - \frac{1}{\delta + 1 + i\varepsilon'} \right\} = -1$$

$$\Rightarrow \quad \frac{1}{2} A_0^a(-\bar{p}) A_0^a(\bar{p}) \left[\bar{p}^2 + m_D^2 \right] \qquad \Rightarrow \text{OK !}$$

(b) Not at all easy, since the integrations $\int \frac{d\Omega}{4\pi} \left\{ \frac{v^\alpha v^\beta}{(v\cdot P)^2} , \frac{v^\alpha v^\beta}{v\cdot P} \right\}$ have a fairly non-trivial structure — however all are known, see e.g. J. Frenkel, J. C. Taylor, Nucl. Phys. B 334 (1990) 199; E. Braaten, R.D. Pisarski, Phys. Rev. D 45 (1992) 1827.

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