Exercise 11

We have seen how to obtain the spectral density, $\rho(\omega)$, from a known $\Pi_E(\omega_n)$, through a certain analytic continuation. There are circumstances, however, where only the function $\tilde{\Pi}_E(\tau) = \langle \hat{A}(\tau)\hat{B}(0)\rangle$ is known, and only numerically (e.g. via a non-perturbative lattice measurement). Show that in this case the following relation holds in between $\rho(\omega)$ and $\tilde{\Pi}_E(\tau)$:

$$\tilde{\Pi}_E(\tau) = \int_0^\infty \frac{d\omega}{\pi}\left[\frac{\rho(\omega)-\rho(-\omega)}{2}\cdot\frac{\cosh\left(\frac{\beta}{2}-\tau\right)\omega}{\sinh\frac{\beta}{2}\omega} + \frac{\rho(\omega)+\rho(-\omega)}{2}\cdot\frac{\sinh\left(\frac{\beta}{2}-\tau\right)\omega}{\sinh\frac{\beta}{2}\omega}\right].$$

Solution

$$\Pi_E(\omega_n) = \int_{-\infty}^{\infty}\frac{d\omega}{\pi}\,\frac{\rho(\omega)}{\omega - i\omega_n} = \int_0^\beta d\tau\, e^{i\omega_n\tau}\,\tilde{\Pi}_E(\tau)$$

Recall: $\quad T\sum_{\omega_n} e^{-i\omega_n\tau} = \delta(\tau \bmod \beta)$.

$$\Rightarrow \tilde{\Pi}_E(\tau) = T\sum_{\omega_n} e^{-i\omega_n\tau}\,\Pi_E(\omega_n)$$

$$= \int_{-\infty}^{\infty}\frac{d\omega}{\pi}\rho(\omega)\cdot T\sum_{\omega_n}\frac{e^{-i\omega_n\tau}}{\omega - i\omega_n}.$$

Let us inspect the sum. Consider separately $\omega<0$ and $\omega>0$:

$\underline{\omega>0}$:

$$T\sum_{\omega_n}\frac{e^{-i\omega_n\tau}}{\omega-i\omega_n} = T\sum_{\omega_n}\int_0^\infty ds\, e^{-i\omega_n\tau - s\omega + si\omega_n}$$

$$= \int_0^\infty ds\, e^{-s\omega}\,\delta(\tau - s \bmod \beta)$$

$$= e^{-\tau\omega}\sum_{n=0}^{\infty} e^{-\beta\omega n} = e^{-\tau\omega}\frac{1}{1-e^{-\beta\omega}}$$

$$= \frac{e^{(\beta-\tau)\omega}}{e^{\beta\omega}-1}. \qquad (0<\tau<\beta)$$

$\underline{\omega<0}$:

$$-T\sum_{\omega_n}\frac{e^{-i\omega_n\tau}}{|\omega|+i\omega_n} = -T\sum_{\omega_n}\int_0^\infty ds\, e^{-i\omega_n\tau - s|\omega| - si\omega_n}$$

$$= -\int_0^\infty ds\, e^{-s|\omega|}\,\delta(\tau+s \bmod \beta)$$

$$= -e^{\tau|\omega|}\sum_{n=1}^{\infty} e^{-\beta|\omega|n} = -\frac{e^{\tau|\omega|}e^{-\beta|\omega|}}{1-e^{-\beta|\omega|}}$$

$$= -\frac{e^{(\beta-\tau)\omega}}{1-e^{\beta\omega}} = \frac{e^{(\beta-\tau)\omega}}{e^{\beta\omega}-1}$$

Now symmetrise this object:

$$f(\omega) = \frac{e^{(\beta-\tau)\omega}}{e^{\beta\omega}-1} = \frac{1}{2}\left[f(\omega)+f(-\omega)\right] + \frac{1}{2}\left[f(\omega)-f(-\omega)\right]$$

Symmetric part:

$$\frac{1}{2}\left[f(\omega)+f(-\omega)\right] = \frac{1}{2}\left\{\frac{e^{(\beta-\tau)\omega}}{e^{\beta\omega}-1} + \frac{e^{(\tau-\beta)\omega}}{e^{-\beta\omega}-1}\right\}$$

$$= \frac{1}{2}\cdot\frac{1}{e^{\beta\omega}-1}\left\{e^{(\beta-\tau)\omega} - e^{\tau\omega}\right\}$$

$$= \frac{1}{2}\,\frac{e^{(\frac{\beta}{2}-\tau)\omega} - e^{(\tau-\frac{\beta}{2})\omega}}{e^{\frac{\beta}{2}\omega} - e^{-\frac{\beta}{2}\omega}}$$

$$= \frac{1}{2}\,\frac{\sinh\left(\frac{\beta}{2}-\tau\right)\omega}{\sinh\left(\frac{\beta}{2}\omega\right)}$$

Antisymmetric part:

$$\frac{1}{2}\left[f(\omega)-f(-\omega)\right] = \frac{1}{2}\left\{\frac{e^{(\beta-\tau)\omega}}{e^{\beta\omega}-1} - \frac{e^{(\tau-\beta)\omega}}{e^{-\beta\omega}-1}\right\}$$

$$= \frac{1}{2}\cdot\frac{1}{e^{\beta\omega}-1}\left\{e^{(\beta-\tau)\omega} + e^{\tau\omega}\right\}$$

$$= \frac{1}{2}\,\frac{\cosh\left(\frac{\beta}{2}-\tau\right)\omega}{\sinh\left(\frac{\beta}{2}\omega\right)}$$

$$\Rightarrow \quad \tilde{\Pi}_E(\tau) = \int_{-\infty}^{\infty}\frac{d\omega}{\pi}\, g(\omega)\, f(\omega)$$

$$= \int_{-\infty}^{\infty}\frac{d\omega}{\pi}\left[\frac{g(\omega)+g(-\omega)}{2} + \frac{g(\omega)-g(-\omega)}{2}\right]\left[\frac{f(\omega)+f(-\omega)}{2} + \frac{f(\omega)-f(-\omega)}{2}\right]$$

Crossterm vanish

$$\doteq \int_{-\infty}^{\infty}\frac{d\omega}{\pi}\left\{\frac{g(\omega)+g(-\omega)}{2}\cdot\frac{1}{2}\frac{\sinh\left(\frac{\beta}{2}-\tau\right)\omega}{\sinh\left(\frac{\beta}{2}\omega\right)} + \frac{g(\omega)-g(-\omega)}{2}\cdot\frac{1}{2}\frac{\cosh\left(\frac{\beta}{2}-\tau\right)\omega}{\sinh\left(\frac{\beta}{2}\omega\right)}\right\}$$

now both term even in $\omega \to -\omega$

$$\doteq \int_{0}^{\infty}\frac{d\omega}{\pi}\left\{\frac{g(\omega)+g(-\omega)}{2}\cdot\frac{\sinh\left(\frac{\beta}{2}-\tau\right)\omega}{\sinh\left(\frac{\beta}{2}\omega\right)} + \frac{g(\omega)-g(-\omega)}{2}\cdot\frac{\cosh\left(\frac{\beta}{2}-\tau\right)\omega}{\sinh\left(\frac{\beta}{2}\omega\right)}\right\} \qquad \square$$

Usually (if $\hat{A} = \hat{B}$) this part vanishes, i.e. $g(\omega)$ is odd.

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