Exercise 6 :

(a) Defining

$$J_f(m,T) = \frac{1}{2} \SumInt_{P_f} \ln(P^2+m^2) \quad ,$$

$$I_f(m,T) = \SumInt_{P_f} \frac{1}{P^2+m^2} \quad ,$$

and writing

$$J_f(m,T) = J_{f,0}(m) + J_{f,T}(m)$$

$$I_f(m,T) = I_{f,0}(m) + I_{f,T}(m) \; ,$$

find the general expressions for

$$J_{f,0}(m) \; , \; I_{f,0}(m), \; J_{f,T}(m), \; I_{f,T}(m) .$$

(b) Find the small-T and large-T expansions for $J_{f,T}(m)$, $I_{f,T}(m)$. Note the absence of odd powers of m in the large-T expansions.

(c) Derive the expression for

$$G_f(\tau) = T \sum_{\omega_n^f} \frac{e^{i\omega_n^f \tau}}{(\omega_n^f)^2 + m^2} .$$

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Solutions:

The general rule : $S_f(T) = 2 S_b\left(\frac{T}{2}\right) - S_b(T)$.

(a) Given the rule, the zero-temperature parts $J_{f,0}(m)$, $I_{f,0}(m)$ obviously remain the same as in the bosonic case.

p. 25: $$J_{b,T}(m) = \int \frac{d^d p}{(2\pi)^d} \, T \ln\left[1 - \exp\left(-\frac{E_p}{T}\right)\right] , \quad E_p = \sqrt{p^2+m^2}$$

$$\Rightarrow \quad J_{f,T}(m) = \int \frac{d^d p}{(2\pi)^d} \, T \left\{ \ln\left[1 - e^{-\frac{2E_p}{T}}\right] - \ln\left[1 - e^{-\frac{E_p}{T}}\right] \right\}$$

$$= \int \frac{d^d p}{(2\pi)^d} \, T \ln\left[1 + e^{-\frac{E_p}{T}}\right]$$

p. 22 : $$I_{b,T}(m) = \int \frac{d^dp}{(2\pi)^d} \frac{1}{E_p} \frac{1}{e^{E_p/T}-1} .$$

p. 60
$$\Rightarrow \quad I_{f,T}(m) = -\int \frac{d^dp}{(2\pi)^d} \frac{1}{E_p} \cdot \frac{1}{e^{E_p/T}+1} .$$

(b)

p. 25 : small-T:
$$J_{b,T}(m) \approx -T^4 \left(\frac{m}{2\pi T}\right)^{\frac{3}{2}} e^{-\frac{m}{T}}$$

$$\Rightarrow J_{f,T}(m) \approx +T^4 \left(\frac{m}{2\pi T}\right)^{\frac{3}{2}} e^{-\frac{m}{T}}$$

$$I_{b,T}(m) \approx \frac{T^3}{m} \left(\frac{m}{2\pi T}\right)^{\frac{3}{2}} e^{-\frac{m}{T}}$$

$$\Rightarrow I_{f,T}(m) \approx -\frac{T^3}{m} \left(\frac{m}{2\pi T}\right)^{\frac{3}{2}} e^{-\frac{m}{T}}$$

large-T:

p. 31: $$J_{b,T}(m) = -\frac{\pi^2 T^4}{90} + \frac{m^2T^2}{24} - \frac{m^3 T}{12\pi} - \frac{m^4}{32\pi^2}\left[\ln \frac{m e^{\gamma_E}}{4\pi T} - \frac{3}{4}\right] + \ldots$$

$$\Rightarrow J_{f,T}(m) = -\frac{1}{8}\frac{\pi^2T^4}{90} + \frac{1}{2}\frac{m^2T^2}{24} - \frac{m^3T}{12\pi} - 2\cdot\frac{m^4}{32\pi^2}\left[\ln\frac{me^{\gamma_E}}{4\pi T} - \frac{3}{4} + \ln 2\right]$$
$$+ \frac{\pi^2T^4}{90} - \frac{m^2T^2}{24} + \frac{m^3T}{12\pi} + \frac{m^4}{32\pi^2}\left[\ln\frac{me^{\gamma_E}}{4\pi T} - \frac{3}{4}\right] + \ldots$$

$$= +\frac{7}{8}\frac{\pi^2T^4}{90} - \frac{m^2T^2}{48} - \frac{m^4}{32\pi^2}\left[\ln\frac{me^{\gamma_E}}{\pi T} - \frac{3}{4}\right] + \ldots$$

the cubic term has disappeared!

p. $\frac{31}{24}$ $\Rightarrow$ $$I_{b,T}(m) = \frac{T^2}{12} - \frac{mT}{4\pi} - \frac{m^2}{8\pi^2}\left[\ln\frac{me^{\gamma_E}}{4\pi T} - \frac{1}{2}\right] + \dots$$

$$\Rightarrow I_{f,T}(m) = \frac{1}{2}\cdot\frac{T^2}{12} - \frac{mT}{4\pi} - 2\cdot\frac{m^2}{8\pi^2}\left[\ln\frac{me^{\gamma_E}}{4\pi T} - \frac{1}{2} + \ln 2\right]$$
$$- \frac{T^2}{12} + \frac{mT}{4\pi} + \frac{m^2}{8\pi^2}\left[\ln\frac{me^{\gamma_E}}{4\pi T} - \frac{1}{2}\right] + \dots$$
$$= -\frac{T^2}{24} - \frac{m^2}{8\pi^2}\left[\ln\frac{me^{\gamma_E}}{\pi T} - \frac{1}{2}\right] + \dots$$

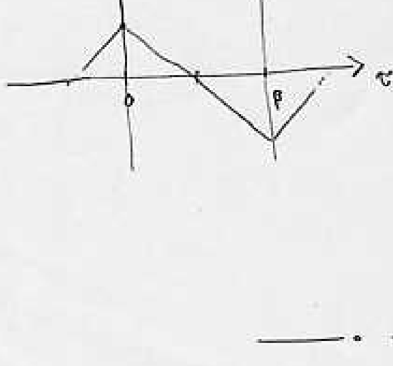
(c) p. 11: $$T\sum_{\omega_n^b}\frac{e^{i\omega_n\tau}}{\omega_n^2+m^2} = \frac{1}{2m}\,\frac{\cosh\left[\left(\frac{\beta}{2}-\tau\right)m\right]}{\sinh\left[\frac{\beta}{2}m\right]}$$

$$= \frac{1}{2m}\,\frac{e^{(\beta-\tau)m}+e^{\tau m}}{e^{\beta m}-1} = \underline{\underline{\frac{1}{2m}\,n_b(m)\left[e^{(\beta-\tau)m}+e^{\tau m}\right]}}$$

$$\Rightarrow T\sum_{\omega_n^f}\frac{e^{i\omega_n\tau}}{\omega_n^2+m^2} = \frac{1}{2m}\left\{\frac{2}{e^{2\beta m}-1}\left[e^{(2\beta-\tau)m}+e^{\tau m}\right] - \frac{1}{e^{\beta m}-1}\left[e^{(\beta-\tau)m}+e^{\tau m}\right]\right\}$$

$$= \frac{1}{2m}\,\frac{1}{(e^{\beta m}-1)(e^{\beta m}+1)}\left\{2e^{(2\beta-\tau)m}+2e^{\tau m} - (e^{\beta m}+1)\left(e^{(\beta-\tau)m}+e^{\tau m}\right)\right\}$$

$$= \frac{1}{2m}\,\frac{1}{(e^{\beta m}-1)}\,\frac{1}{(e^{\beta m}+1)}\left\{\underbrace{2e^{(2\beta-\tau)m}+2e^{\tau m} - e^{(2\beta-\tau)m} - e^{(\beta+\tau)m} - e^{(\beta-\tau)m} - e^{\tau m}}_{e^{(2\beta-\tau)m}+e^{\tau m}-e^{(\beta+\tau)m}-e^{(\beta-\tau)m} \,=\, (e^{\beta m}-1)\left(e^{(\beta-\tau)m}-e^{\tau m}\right)}\right\}$$



$$= \underline{\underline{\frac{1}{2m}\,n_f(m)\left[e^{(\beta-\tau)m}-e^{\tau m}\right]}}$$