Exercise 3:

(a) Complete the result for $I(m,T)$ to order m^4.

(b) Use the result to derive $J(m,T)$ to order m^6.

(c) Subtracting $J_0(m)$, verify that $J_T(m)$ is finite in the limit $\varepsilon \to 0$.

(d) Evaluate $J_T(m)$ numerically, and sketch the validity regions of the small-T and large-T expansions.

— o —

Solutions:

(a)

* Zero-mode from p. 27: $-\frac{mT}{4\pi}$

* $l=0$:

$$2T \, \frac{1}{(4\pi)^{3/2}} (2\pi T) \cdot \frac{-2\sqrt{\pi}}{1} \cdot \left(-\frac{1}{12}\right) = \frac{T^2}{12}$$

* $l=1$:

$$2T \cdot \frac{1}{(4\pi)^{3/2}} \cdot (4\pi)^{\varepsilon} \cdot (2\pi T) \, (2\pi T)^{-2\varepsilon} \cdot \left(\frac{-m^2}{(2\pi T)^2}\right) \cdot$$

$$\cdot \sqrt{\pi} \left(1 - \varepsilon\gamma_E - 2\varepsilon \ln 2\right) \cdot \frac{1}{2\varepsilon} \left(1 + 2\varepsilon\gamma_E\right)$$

$$= -\frac{m^2}{(4\pi)^2} \left[\frac{1}{\varepsilon} + \ln 4\pi \; \substack{-2\ln 2\pi \\ -2\ln 2} \; - \ln T^2 \; \substack{+2\gamma_E \\ -\gamma_E} \right]$$

$$= -\frac{m^2 \mu^{-2\varepsilon}}{(4\pi)^2} \left[\frac{1}{\varepsilon} + \ln \frac{\mu^2}{T^2} + \ln 4\pi - \gamma_E + 2(\gamma_E - \ln 4\pi) \right]$$

p. 24

$$= -\frac{m^2}{(4\pi)^2} \mu^{-2\varepsilon} \left[\frac{1}{\varepsilon} + \ln \frac{\bar{\mu}^2}{T^2} + 2(\gamma_E - \ln 4\pi) \right]$$

* $l=2$: $$2T \cdot \frac{1}{(4\pi)^{3/2}} \cdot 2\pi T \cdot \frac{m^4}{(2\pi T)^4} \cdot \frac{\frac{1}{2}\sqrt{\pi}}{2} \cdot \zeta(3)$$

$$= \frac{m^4 \, \zeta(3)}{128 \pi^4 T^2}$$

$$\Rightarrow \quad I(m,T) = \frac{T^2}{12} - \frac{mT}{4\pi} - \frac{m^2}{(4\pi)^2}\mu^{-2\epsilon}\left[\frac{1}{\epsilon} + \ln\left(\frac{\bar{\mu}e^{\gamma_E}}{4\pi T}\right)^2\right] + \frac{m^4 \zeta(3)}{128\pi^4 T^2} + O(m^6).$$

(b) p. 22: $\quad m \cdot I(m,T) = \frac{dJ(m,T)}{dm}$

p. 26: $\quad J(0,T) = -\frac{\pi^2 T^4}{90}$

$$\Rightarrow \quad J(m,T) = -\frac{\pi^2 T^4}{90} + \frac{m^2 T^2}{24} - \frac{m^3 T}{12\pi} - \frac{m^4}{64\pi^2}\mu^{-2\epsilon}\left[\frac{1}{\epsilon} + \ln\left(\frac{\bar{\mu}e^{\gamma_E}}{4\pi T}\right)^2\right]$$
$$+ \frac{m^6 \zeta(3)}{768\pi^4 T^2} + O(m^8)$$

the famous "cubic" term. The only one which is non-analytic in m^2, and extremely important!

(c) p. 24: $\quad J_0(m) = -\frac{m^4}{64\pi^2}\mu^{-2\epsilon}\left[\frac{1}{\epsilon} + \ln\frac{\bar{\mu}^2}{m^2} + \frac{3}{2}\right]$

$$\Rightarrow \quad J_T(m) = J(m,T) - J_0(m)$$
$$= -\frac{\pi^2 T^4}{90} + \frac{m^2 T^2}{24} - \frac{m^3 T}{12\pi} - \frac{m^4}{64\pi^2}\left[\ln\left(\frac{m e^{\gamma_E}}{4\pi T}\right)^2 - \frac{3}{2}\right] + \frac{m^6\zeta(3)}{768\pi^4 T^2} + \ldots$$

finite!

> Another exercise: obtain the same result using cutoff regularisation, checking for instance that the linear divergence $\frac{T\Lambda}{2\pi^2}$ in $I^{(n=0)}$ cancels!!

(d) Consider

$$\frac{J_T(m)}{T^4} = \frac{1}{2\pi^2}\int_0^\infty dx\, x^2 \ln\left[1 - e^{-\sqrt{x^2+y^2}}\right]_{y=\frac{m}{T}} \equiv \mathcal{J}(y)$$

$$\underset{p\,25}{\overset{y\gg 1}{=}} \; -\left(\frac{y}{2\pi}\right)^{\frac{3}{2}} e^{-y}$$

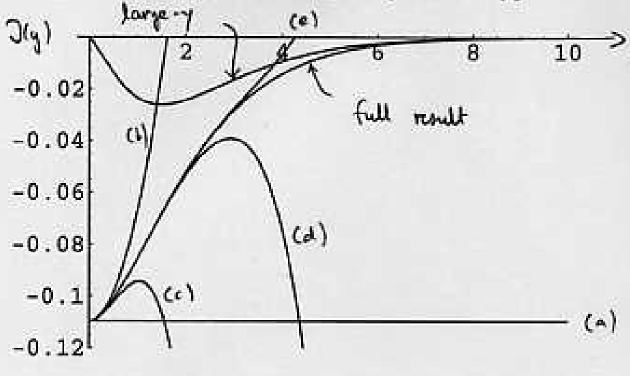
$$\underset{p.\,31}{\overset{y\ll 1}{=}} \; \underset{(a)}{-\frac{\pi^2}{90}} + \underset{(b)}{\frac{y^2}{24}} - \underset{(c)}{\frac{y^3}{12\pi}} \underset{(d)}{- \frac{y^4}{32\pi^2}\left[\ln y + \gamma_E - \ln(4\pi) - \frac{3}{4}\right]} + \underset{(e)}{\frac{y^6\,\zeta(3)}{768\,\pi^4}} + \ldots$$

```
In[23]:= full[y_] :=
          1 / 2 / Pi ^ 2 NIntegrate[x ^ 2 Log[1 - Exp[-Sqrt[x ^ 2 + y ^ 2]]], {x, 0, Infinity}]

In[24]:= lowT[y_] := -Exp[-y] (y / 2 / Pi) ^ {3 / 2}

In[25]:= high1[y_] := -Pi ^ 2 / 90
         high2[y_] := -Pi ^ 2 / 90 + y ^ 2 / 24
         high3[y_] := -Pi ^ 2 / 90 + y ^ 2 / 24 - y ^ 3 / 12 / Pi
         high4[y_] := -Pi ^ 2 / 90 + y ^ 2 / 24 - y ^ 3 / 12 / Pi -
           y ^ 4 / 32 / Pi ^ 2 (Log[y] + EulerGamma - Log[4 Pi] - 3 / 4)
         high5[y_] := -Pi ^ 2 / 90 + y ^ 2 / 24 - y ^ 3 / 12 / Pi -
           y ^ 4 / 32 / Pi ^ 2 (Log[y] + EulerGamma - Log[4 Pi] - 3 / 4) + y ^ 6 Zeta[3] / 768 / Pi ^ 4

In[30]:= Plot[{full[y], lowT[y], high1[y], high2[y], high3[y], high4[y], high5[y]},
          {y, 0.01, 10}, PlotRange -> {-0.12, 0}]
```




```
Out[30]= - Graphics -
```