Exercise 1:

(a) Defining $\hat{x}(\tau) \equiv e^{\hat{H}\tau}\,\hat{x}\,e^{-\hat{H}\tau}$, $\tau > 0$, show that the path integral expression for

$$G(\tau) \equiv \frac{\mathrm{Tr}\left[e^{-\beta\hat{H}}\,\hat{x}(\tau)\,\hat{x}(0)\right]}{\mathrm{Tr}\left[e^{-\beta\hat{H}}\right]}$$

reads

$$G(\tau) = \frac{\int \mathcal{D}x \;\; x(\tau)\,x(0)\;\exp\left[-\frac{1}{\hbar}S_E\right]}{\int \mathcal{D}x\;\exp\left[-\frac{1}{\hbar}S_E\right]}$$

(b) Compute $G(\tau)$ for the harmonic oscillator, either directly [expressing $\hat{H}, \hat{x}$ in terms of $\hat{a}, \hat{a}^{+}$], or using the path integral.

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Solution:

(a) Straightforward ...

(b)

* Integration measure: $C' \int_{-\infty}^{\infty} dx_0 \int_{-\infty}^{\infty} \prod_{n>0} da_n db_n$

* Exponential: $\exp\left[-\frac{mT}{2}\omega^2 x_0^2 - mT\sum_{n>0}(\omega_n^2+\omega^2)(a_n^2+b_n^2)\right]$

* Observable: $\left(a_{-p}=a_p,\; b_{-p}=-b_p\right)$

$$x(\tau) = Tx_0 + T\sum_{p>0}\left[(a_p+ib_p)\,e^{i\omega_p\tau} + (a_p-ib_p)\,e^{-i\omega_p\tau}\right]$$

$$x(0) = T\left[x_0 + \sum_{r>0} 2a_r\right]$$

$$\Rightarrow \quad G(\tau) = \frac{\int dx_0 \int \prod_{n>0} da_n db_n \;\; x(\tau)\,x(0)\;\exp[\cdots]}{\int dx_0 \int \prod_{n>0} da_n db_n \;\exp[\cdots]} \equiv \langle x(\tau)\,x(0)\rangle$$

Since exponential is quadratic in $x_0, a_n, b_n \in \mathbb{R}$, we have

$$\langle x_0 a_p\rangle = \langle x_0 b_p\rangle = \langle a_p b_r\rangle = 0$$

$$\langle a_p a_r\rangle = \langle b_p b_r\rangle \propto \delta_{pr}$$

$$\Rightarrow \quad G(\tau) = T^2 \left\langle x_0^2 + \sum_{p>0}\sum_{r>0} 2a_r\, a_p \left(e^{i\omega_p\tau} + e^{-i\omega_p\tau}\right) \right\rangle$$

$$= T^2 \left\langle x_0^2 + \sum_{p>0} 2a_p^2 \left(e^{i\omega_p\tau} + e^{-i\omega_p\tau}\right) \right\rangle$$

Here: $$\langle x_0^2 \rangle = \frac{\int dx_0\, x_0^2 \exp\left(-\frac{mT}{2}\omega^2 x_0^2\right)}{\int dx_0 \exp\left(-\frac{mT}{2}\omega^2 x_0^2\right)} = -\frac{2}{m\omega^2}\frac{d}{dT}\left[\ln \int dx_0 \exp\left(-\frac{mT}{2}\omega^2 x_0^2\right)\right]$$

$$= -\frac{2}{m\omega^2}\frac{d}{dT}\left[\ln\sqrt{\frac{2\pi}{mT\omega^2}}\right]$$

$$= -\frac{2}{m\omega^2}\frac{d}{dT}\ln T^{-\frac{1}{2}} = \frac{1}{m\omega^2 T}$$

$$\langle a_p^2 \rangle = \frac{\int da_p\, a_p^2 \exp\left(-mTa_p^2(\omega_p^2+\omega^2)\right)}{\int da_p \exp\left(-mTa_p^2(\omega_p^2+\omega^2)\right)} = \frac{1}{2m(\omega_p^2+\omega^2)T}$$

$$\Rightarrow \quad G(\tau) = \frac{T}{m}\left(\frac{1}{\omega^2} + \sum_{p>0}\frac{e^{i\omega_p\tau}+e^{-i\omega_p\tau}}{\omega_p^2+\omega^2}\right) = \frac{1}{m}\cdot T\sum_{p=-\infty}^{\infty}\frac{e^{i\omega_p\tau}}{\omega_p^2+\omega^2}, \quad \omega_p = \frac{2\pi n T}{\hbar}$$

How to compute the sum? There are various ways, here's one:

(a) Note that $\left(-\frac{d^2}{d\tau^2}+\omega^2\right)G(\tau) = \frac{1}{m}T\sum_{p=-\infty}^{\infty} e^{i\omega_p\tau}$

$$\sum_{p=-\infty}^{\infty} e^{ixp} = A\,\delta(x \bmod 2\pi) \quad \Big| \int_{-\varepsilon}^{2\pi-\varepsilon} dx$$

$$\underbrace{\sum_{p=-\infty}^{\infty}\frac{1}{ip}\left[e^{i(2\pi-\varepsilon)p} - e^{i(-\varepsilon)p}\right]}_{\substack{0,\ p\neq 0 \\ 2\pi,\ p=0}} = A$$

$$\Rightarrow \sum_{p=-\infty}^{\infty} e^{ixp} = 2\pi\,\delta(x \bmod 2\pi)$$

$$\sum_{p=-\infty}^{\infty} e^{ip\frac{2\pi T\tau}{\hbar}} = \beta\hbar\,\delta(\tau \bmod \beta\hbar)$$

$$\Rightarrow \boxed{\left(-\frac{d^2}{d\tau^2}+\omega^2\right)G(\tau) = \frac{\hbar}{m}\,\delta(\tau \bmod \beta\hbar)}$$

Let us look at $0 < \tau < \beta\hbar$.

$$\Rightarrow \quad \left(-\frac{d^2}{d\tau^2} + \omega^2\right) G(\tau) = 0$$

$$\Rightarrow \quad G(\tau) = A \cdot e^{\omega\tau} + B\, e^{-\omega\tau}$$

From the definition, $G(\beta\hbar - \tau) = G(\tau)$

$$A e^{\omega\beta\hbar} e^{-\omega\tau} + B e^{-\omega\beta\hbar} e^{\omega\tau} = A e^{\omega\tau} + B e^{-\omega\tau}$$

$$\Rightarrow \quad A e^{\omega\beta\hbar} = B$$

$$\Rightarrow \quad G(\tau) = A\left[e^{\omega\tau} + \omega^{\omega(\beta\hbar - \tau)}\right]$$

To figure out A, we can look at $\tau = 0$.

$$G(0) = A\left(1 + e^{\omega\beta\hbar}\right) = \frac{1}{m} \cdot T \sum_{p=-\infty}^{\infty} \frac{1}{\omega^2 + \omega_p^2}$$

$$= \frac{T}{m} \cdot \frac{\hbar^2}{(2\pi T)^2} \underbrace{\sum_{p=-\infty}^{\infty} \frac{1}{\left(\frac{\hbar\omega}{2\pi T}\right)^2 + p^2}}$$

Mathematica:

$$\frac{\pi \cdot \cosh\left(\frac{\hbar\omega}{2T}\right)}{\frac{\hbar\omega}{2\pi T} \sinh\left(\frac{\hbar\omega}{2T}\right)}$$

$$= \frac{1}{2m} \cdot \frac{\hbar}{\omega} \cdot \frac{\cosh\left(\frac{\hbar\omega}{2T}\right)}{\sinh\left(\frac{\hbar\omega}{2T}\right)}$$

$$\Rightarrow A = \frac{\hbar}{m} \cdot \frac{1}{2\omega} \cdot \frac{e^{\frac{\hbar\omega\beta}{2}} + e^{-\frac{\hbar\omega\beta}{2}}}{e^{\frac{\hbar\omega\beta}{2}} - e^{-\frac{\hbar\omega\beta}{2}}} \cdot \frac{1}{e^{\hbar\omega\beta} + 1} = \frac{\hbar}{m} \cdot \frac{e^{-\frac{\hbar\omega\beta}{2}}}{2\omega} \cdot \frac{1}{e^{\frac{\hbar\omega\beta}{2}} - e^{-\frac{\hbar\omega\beta}{2}}}$$

$$\Rightarrow G(\tau) = \frac{\hbar}{m} \cdot \frac{e^{\omega\left(\tau - \frac{\beta\hbar}{2}\right)} + e^{\omega\left(\frac{\beta\hbar}{2} - \tau\right)}}{2\omega\left(e^{\frac{\hbar\omega\beta}{2}} - e^{-\frac{\hbar\omega\beta}{2}}\right)} = \boxed{\frac{\hbar}{m} \cdot \frac{1}{2\omega} \cdot \frac{\cosh\left[\left(\frac{\beta\hbar}{2} - \tau\right)\omega\right]}{\sinh\left[\frac{\beta\hbar\omega}{2}\right]}}$$

discontinuity of derivative $= \frac{\hbar}{m}$