3.2 Topological gauge configurations (non-Abelian instantons, sphalerons)

Outline: In sec. 2.3, we have defined an instanton configuration as a solution of Euclidean equations of motion, and argued that it describes quantum tunnelling.
In sec. 3.1, we have considered gauge field theories, and argued that they contain extended "solitonic" objects, non-trivial solutions of time-independent equations of motion.
Now we consider solutions having ingredients from both classes: related to the physics of tunnelling, and carrying topological properties, yet not necessarily stable like solitons, but just saddle points.

Vacuum topology: We consider a (1+3)-dimensional Euclidean spacetime, in the sense that was introduced in secs. 2.2 and 2.3, and a corresponding Euclidean action, S_E. The theory may contain matter fields ϕ (c.f. p. 51), but more importantly it has gauge fields A_μ, associated with a group G. (We assume that $G = SU(2)$, or that $SU(2)$ is a subgroup of G.

Page 4: as a manifold, $SU(2)$ is like S^3.

Let us consider a "hypersphere" of a large radius,

$$R \equiv \sqrt{\tau^2 + |\vec{x}|^2} .$$

This is also S^3, by definition.*

* Here we assume $\beta\hbar \to \infty$, like on p. 43, and can also make the τ-range symmetric, $(-\frac{\beta\hbar}{2}, \frac{\beta\hbar}{2})$. We return to a finite $\beta\hbar$ on p. 51.

If we now move along the hypersphere, and carry out a gauge transformation of the trivial** vacuum at each point,

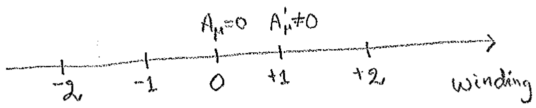
$$A'_\mu(\tau,\vec{x}) = \underbrace{U(\tau,\vec{x})\overbrace{A_\mu(\tau,\vec{x})}^{0}U^+(\tau,\vec{x})}_{0} + \frac{i}{g}\underbrace{U(\tau,\vec{x})}_{\in SU(2)}\partial_\mu U^+(\tau,\vec{x}) ,$$

p. 45

** Trivial means vanishing gauge potentials, $A_\mu = 0$.

We obtain a mapping $S^3 \to S^3$, $(\tau,\vec{x}) \mapsto U(\tau,\vec{x})$. As before with vortices (p. 47) and monopoles (p. 48), this mapping defines a non-trivial homotopy group: $\pi_3(S^3) \cong \mathbb{Z}$.

Most importantly, if the mapping has non-trivial winding $\neq 0$, it cannot be continuously deformed to vanishing winding. So, there appear to be $\mathbb{Z}$ degenerate vacua!



A concrete example of a winding gauge transformation:

$$U(\tau,\vec{x}) = \frac{\tau\sigma_0 + i\vec{x}\cdot\vec{\sigma}}{R} \qquad \text{(c.f. p. 4)} ,$$

$$U^+U = \frac{(\tau\sigma_0 - i\vec{x}\cdot\vec{\sigma})(\tau\sigma_0 + i\vec{x}\cdot\vec{\sigma})}{R^2} = \frac{\tau^2 + \vec{x}^2}{R^2}\sigma_0 = \mathbb{1}_{2\times 2} ,$$

$$\det U = \frac{\tau^2 + \vec{x}^2}{R^2} = 1 .$$

This "points outwards" at every location of the hypersphere.

Instantons:

The situation with apparently degenerate vacua resembles that on p. 42. So we may ask if there are "saddle points", solutions of the Euclidean equations of motion, that describe tunnellings between the vacua?

Like with the "kink" solution from exercise 11.3, the gauge field should be in one vacuum in some domain (say, $A_\mu = 0$ for $R \to 0$), and in another far away (say, $A_\mu = \frac{i}{g} U(\tau,\vec{x})\partial_\mu U^\dagger(\tau,\vec{x})$, with $U = \frac{\tau\sigma_0 + i\vec{x}\cdot\vec{\sigma}}{R}$ for $R \to \infty$). In these asymptotic domains, $\mathcal{L}_E = 0$, so that the integral $S_E = \int d\tau \int d^3\vec{x}\, \mathcal{L}_E$ is finite.

In between the vacua, the gauge field cannot be of "pure gauge" form, because of the topological reason explained on p. 49. So, $\mathcal{L}_E \neq 0$, and correspondingly the instanton action, $\bar{S}_E$, is positive.

Can such configurations be found? Yes! *
Let us mention three of their important characteristics:

* the saddle point action is** $\bar{S}_E = \frac{8\pi^2}{g^2}|n|$ (winding) (there may be other saddle points with larger actions). For a weak coupling, $g^2\hbar \ll 1$, the weight (cf. p. 44)
$$e^{-\frac{\bar{S}_E}{\hbar}} = e^{-\frac{8\pi^2}{g^2\hbar}}$$
is exponentially small. Therefore, we may expect tunnellings to happen very rarely.

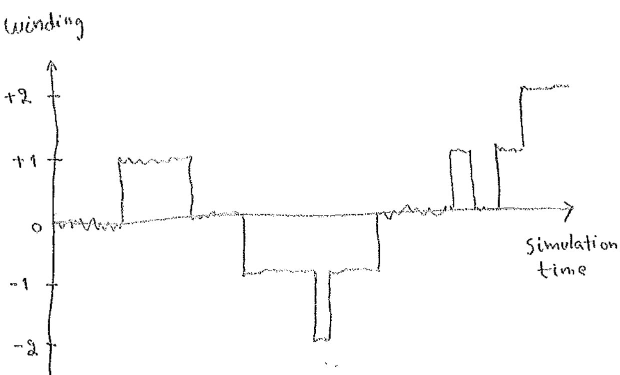
* however, in pure Yang-Mills theory, it turns out that there is not one but infinitely many instanton configurations, labelled by a continuous parameter, called the "instanton size" (ρ).

  The problem is that the integral over ρ (similar in spirit to the zero-mode integral on p. 44) is divergent! Therefore, the saddle point approximation does not yield a well-defined result for pure Yang-Mills theory at $\beta\hbar \to \infty$.

* despite the problem, the existence of distinct topological vacua and the difficulty of tunnelling between them, if $g^2\hbar$ is small, are very much visible, if the path integral for pure Yang-Mills theory is computed numerically with so-called lattice Monte Carlo simulations.

* A. A. Belavin, A. M. Polyakov, A. S. Schwartz, Yu. S. Tyupkin, "Pseudoparticle Solutions of the Yang-Mills Equations", Phys. Lett. B 59 (1975) 85.

** G. 't Hooft, "Computation of quantum effects due to a four-dimensional pseudoparticle", Phys. Rev. D 14 (1976) 3432.

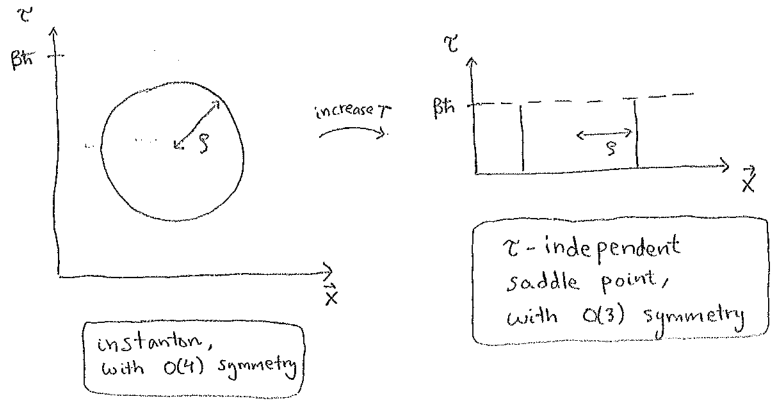


Extensions: We now generalize the instanton considerations in two respects:

* we keep $\beta\hbar$ finite, in fact we make it small, which corresponds to a high temperature.
* we add a field ϕ, transforming under the fundamental representation of $SU(2)$, so that $\mathcal{L}_E$ looks like on p. 46:

$$\mathcal{L}_E \supset (D_\tau\phi)^\dagger(D_\tau\phi) + (D_i\phi)^\dagger(D_i\phi) + \lambda(\phi^\dagger\phi - a^2)^2.$$

High temperature: If $\beta\hbar$ is small, Euclidean spacetime is $O(3)$ rather than $O(4)$ symmetric. Moreover, we should remember that the τ-direction is "periodic" (p.37): $\phi(0,\vec{x}) = \phi(\beta\hbar,\vec{x})$.



Two important consequences from small $\beta\hbar$:

* the mapping onto vacuum gauge configurations at large distances is now from S^2 onto S^3. There is no homotopy associated with this mapping; therefore, such configurations are not topologically stable. They are also not local minima of S_E, just extrema.
* because of time independence, the weight becomes

$$e^{-\frac{S_E}{\hbar}} = e^{-\frac{1}{\hbar}\int_0^{\beta\hbar} d\tau \int d^3\vec{x}\, \mathcal{L}_E} = e^{-\beta\int d^3\vec{x}\, \mathcal{L}_E}.$$

This is like a Boltzmann weight. Therefore, the τ-independent configurations correspond to classical thermal fluctuations, rather than quantum tunnelling.

Role of ϕ: Unlike with monopoles (p.48), there is no topology associated with the vacuum manifold of ϕ: the little group H is trivial, and $G/H \sim S^3$. However, the mass scales that ϕ brings along ("Higgs mechanism") effectively fix the size ρ (cf. p.50), so that the divergence affecting instantons is lifted.

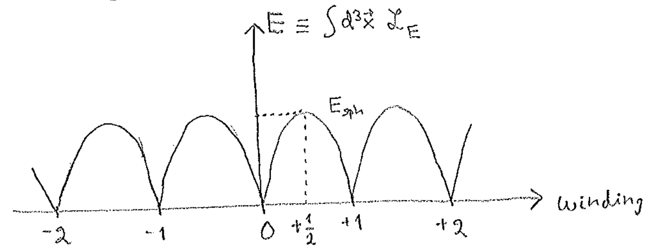
Sphalerons:

An explicit τ-independent solution, called a "sphaleron" ("ready to fall"), can be constructed numerically.*
Some of its main characteristics are:

* to guarantee finite energy, gauge fields should asymptotically be of "pure gauge" type, i.e. $A_k \approx \frac{i}{g} U \partial_k U^+$, with now $U = \frac{i \vec{x} \cdot \vec{\sigma}}{r}$, modified perhaps by a global ($\vec{x}$-independent) transformation.
* the "energy" of the solution is
$$E_{sph} \equiv \int d^3\vec{x}\, \bar{\mathcal{L}}_E = \frac{8\pi m_W}{g^2} f(\lambda) ,$$
where m_W is the mass of $W^\pm$-bosons, and $f(\lambda) \sim 1$ is a slowly varying function.
* in terms of the "winding" sketched on p. 49, the sphaleron has the value $+\frac{1}{2}$.

* F.R. Klinkhamer and N.S. Manton, "A saddle-point solution in the Weinberg-Salam theory", Phys. Rev. D 30 (1984) 2212.

Therefore we can now draw an energy landscape:



Physics:

It turns out that the sphaleron transitions, if they happen, lead to a very important consequence: they violate baryon + lepton number (B+L).**

Under terrestrial circumstances, B+L violation has never been observed. But the universe as a whole has a "baryon asymmetry" (very many baryons, very few antibaryons). Perhaps sphaleron processes taking place at temperatures $k_B T \gg E_{sph}$ could explain this? This is a very important topic for cosmology.***

** G. 't Hooft, "Symmetry breaking through Bell-Jackiw anomalies", Phys. Rev. Lett. 37 (1976) 8.

*** V.A. Kuzmin, V.A. Rubakov, M.E. Shaposhnikov, "On the anomalous electroweak baryon number non-conservation in the Early Universe", Phys. Lett. B 155 (1985) 36.