3. Kind of a synthesis

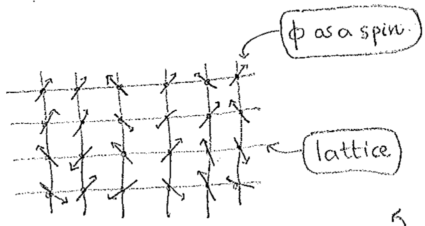
3.1 Topological defects (vortices, monopoles)

Outline: In the previous section we have discussed Minkowskian and Euclidean path integrals for one degree of freedom (ϕ). We now generalize to field theory, where there is ϕ at each spatial location, and $\phi \in V$ transforms under some representation of a group G. Then we will find quite interesting solutions of the field equations.

Fields and symmetries:

$$\phi(t) \to \phi(\vec{x},t)$$

$$S_M \to \int dt \int d^3\vec{x}\, \mathcal{L}_M$$



One can think of the fields a bit like spins on a lattice.

The fields transform under some representation of the Lorentz group (cf. sec. 1.7), but $\mathcal{L}_M$ should be invariant, i.e. in the "trivial" $(0,0)$ representation of $L^{\uparrow}_{+}$.

In addition, the fields may transform under some "internal symmetry", or group G. Most interesting to us is a "local symmetry", where there is a separate transformation at each $\vec{x}$. We denote the transformation by $D(g(\vec{x})) \equiv U(\vec{x})$, because in particle physics, "g" usually denotes a coupling constant (see below), rather than a group element. So, we transform

$$\phi(\vec{x}) \to \phi'(\vec{x}) = U(\vec{x})\phi(\vec{x}).$$

Gauge fields: If $U(\vec{x})$ is unitary, it is easy to build some invariants, like $\phi^\dagger(\vec{x})\phi(\vec{x})$. But this does not work nicely with derivatives:

$$\partial_\mu \phi(\vec{x}) \to \partial_\mu\{U(\vec{x})\phi(\vec{x})\} = U(\vec{x})\{\underbrace{U^\dagger(\vec{x})\partial_\mu U(\vec{x})}_{\text{"extra"}}\phi(\vec{x}) + \partial_\mu\phi(\vec{x})\}.$$

To compensate for this, we introduce a "covariant derivative":

$$D_\mu\phi(\vec{x}) \equiv \{\partial_\mu - i\underbrace{g}_{\text{"coupling constant"}}\underbrace{A_\mu(\vec{x})}_{\text{"gauge field"}}\}\phi(\vec{x}).$$

Then, if $A'_\mu(\vec{x}) \equiv U(\vec{x})A_\mu(\vec{x})U^\dagger(\vec{x}) + \frac{i}{g}U(\vec{x})\partial_\mu U^\dagger(\vec{x})$,

$$D'_\mu\phi'(\vec{x}) = U(\vec{x})\{\partial_\mu + \underbrace{U^\dagger\partial_\mu U}_{\text{from }\partial_\mu} - igA_\mu + \underbrace{\partial_\mu U^\dagger U}_{\text{from gauge field}}\}\phi(\vec{x})$$

$$= U(\vec{x})D_\mu\phi(\vec{x}).$$

$$\boxed{U^\dagger\partial_\mu U + \partial_\mu U^\dagger U = \partial_\mu(U^\dagger U) = \partial_\mu \mathbb{1} = 0.}$$

Higgs mechanism: We consider the following Lagrangian, invariant under both $L^\uparrow_+$ and G:

$$\mathcal{L}_M = -\eta^{\mu\nu}(D_\mu\phi)^\dagger(D_\nu\phi) - \lambda(\phi^\dagger\phi - a^2)^2 ,$$

$$\mathcal{L}_E = (D_\tau\phi)^\dagger(D_\tau\phi) + (D_i\phi)^\dagger(D_i\phi) + \lambda(\phi^\dagger\phi - a^2)^2 .$$

p.37: $\mathcal{L}_E = -\mathcal{L}_M(t=-i\tau)$

The Lagrangian $\mathcal{L}_E$, and the action $S_E = \int_0^{\beta\hbar} d\tau \int d^3\vec{x}\, \mathcal{L}_E$, are minimized if $\phi^\dagger\phi = a^2$; this is called the Higgs mechanism.

Defects: The solution of the condition $\phi^\dagger\phi = a^2$ is not unique. Let us call ϕ_0 one solution. Then the gauge transform $\phi_0'(\vec{x}) = U(\vec{x})\phi_0$ is also a solution $\forall\, \vec{x}$, but the value varies from point to point. However, the variations lead to derivatives, $\partial_i\phi_0'(\vec{x})$. A "defect" is a solution which is:

* time-independent : $D_\tau\phi_0' = \partial_\tau\phi_0' = 0$.

  in gauge $A_0 = 0$

* carries finite "energy" : $\int_{\text{all space}} d^d\vec{x}\, \mathcal{L}_E < \infty$.

To minimize the energy, the solutions tend to have maximal symmetry, e.g. cylindrical symmetry (vortex, p.47, $d=2$) or spherical symmetry (monopole, p.48; $d=3$).

Some group theory: Before going to the concrete constructions, let us summarize the group theory pertinent to the problem at hand:

* the overall gauge group is G
* we call H the subgroup that leaves ϕ_0 invariant:

$$U_h \in H \iff U_h\phi_0 = \phi_0 .$$

* the left coset space G/H then corresponds to the different independent "directions" of ϕ_0 (cf. p.30,31).
* the coset space G/H may or may not allow for a group structure (cf. p.31). But this is not important for the topic of the present lecture.
* consider mappings $[0,1]^n \to G/H$, where the boundary is mapped onto a single point. In words:
  - $n=1 \iff$ draw a closed loop on G/H;
  - $n=2 \iff$ draw a closed surface on G/H.
* a homotopy group $\pi_n(G/H)$ characterizes these mappings. The value $\in \mathbb{Z}$ is "0" if the result is contractible; ± 1 if it wraps around once, etc. Concrete examples will follow.

Vortices:

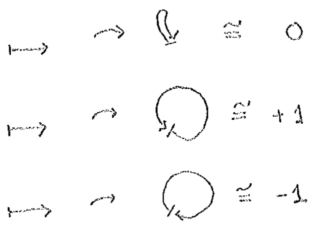
In this case, $\phi \in \mathbb{C}$ and $G = U(1)$.
[Or, equivalently, $\phi \in \mathbb{R}^2$ and $G = SO(2)$].

If we choose $\phi_0 = a \in \mathbb{R}$, there is no non-trivial phase which leaves this invariant, so $H \cong \{e\}$.

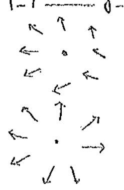
Then $G/H = U(1)$, which corresponds as a manifold to a circle, S^1.

Homotopy: $\pi_1(S^1) \cong \mathbb{Z}$.

mathematically:



physically:



As formulae: $\phi(\vec{x}) = a e^{i\Theta(\vec{x})}$; $\vec{x} = \underbrace{|\vec{x}|}_{\text{distance from center}} e^{i\varphi}$

$$\Rightarrow \quad n = \int_0^{2\pi} \frac{d\varphi}{2\pi} \frac{d\Theta}{d\varphi} \in \mathbb{Z}. \qquad \text{"winding number"}$$

Energetics:

Consider now the energy per unit length of a vortex:

$$\frac{\Delta E}{\Delta z} = \int_{(z=\text{const.})} d^2\vec{x}\, \mathcal{L}_E \quad ; \quad \vec{x} = r(\cos\varphi, \sin\varphi) \, ; \quad r = |\vec{x}| .$$

If $\phi_0 = a e^{i\Theta(\varphi)}$, then $\phi_0^\dagger \phi_0 - a^2 = 0$.

Also, we can assume $D_r \phi_0 = (\partial_r - igA_r)\phi_0 = -igA_r\phi_0$ to decay fast as $r \to \infty$, if A_r does so.

But how about the angular direction?

$$D_\varphi \phi_0 = \left[\frac{1}{r}\partial_\varphi - igA_\varphi\right]\phi_0 = \left[\frac{1}{r} i \frac{d\Theta}{d\varphi} - igA_\varphi\right]\phi_0 .$$

If $A_\varphi = 0$, $(D_\varphi\phi_0)^\dagger (D_\varphi \phi_0) = \frac{a^2}{r^2}\left(\frac{d\Theta}{d\varphi}\right)^2$, and $\int_{r_0}^{\infty} dr\, r \frac{1}{r^2}$ diverges!↯

But if $A_\varphi \approx \frac{1}{gr}\frac{d\Theta}{d\varphi}$, the integral is <u>finite</u>! This we require.

Flux:

Let us compute the magnetic flux through a vortex:

$$\Phi_B = \int \underbrace{d^2\vec{x}\, \vec{e}_z}_{d\vec{S}} \cdot \vec{B} = \int d\vec{S} \cdot \nabla \times \vec{A} \underset{\text{Stokes}}{=} \oint d\vec{r} \cdot \vec{A}$$

$$\underset{d\vec{r} = r\,d\varphi\,\vec{e}_\varphi}{=} \int_0^{2\pi} r\,d\varphi\, A_\varphi \underset{\text{from above}}{=} \frac{1}{g}\int_0^{2\pi} d\varphi \frac{d\Theta}{d\varphi} = \frac{2\pi n}{g} .$$

So, the magnetic flux appears to be quantized!

Monopoles: Finally we outline what happens with the famous 't Hooft – Polyakov monopoles.* In this case the gauge group is $SU(2)$, and the field ϕ transforms in the adjoint representation (cf. p.10). According to exercise 1.4, this means that the field transformation corresponds to $SO(3)$, i.e. rotations.

* A.M. Polyakov, "Particle Spectrum in Quantum Field Theory", JETP Lett. 20 (1974) 194; G. 't Hooft, "Magnetic Monopoles in Unified Gauge Theories," Nucl. Phys. B 79 (1974) 276.

The little group H is then those rotations which leave ϕ_0, e.g. $\phi_0 = (0,0,a)$, invariant : this is $H \cong SO(2)$ (cf. p. 31). The coset space (not group) is $SO(3)/SO(2) \sim S^2$, a sphere.

Now the interesting homotopy consists of covering the sphere with "sheets", $[0,1]^2$. The group is non-trivial, corresponding to the number of "wrappings": $\pi_2(S^2) \cong \mathbb{Z}$. Monopoles are solutions in which ϕ_0' wraps non-trivially around the origin.

Energetics: Like with vortices, it is important that if ϕ_0' depends on angles at large distances, the corresponding contribution to energy still remains finite. Let us demonstrate how this goes.

* consider $\phi_0 \in \mathbb{R}^3$ with components $\phi_{0,b} = \frac{x_b}{r}\, a$. Then $|\phi_0|^2 = a^2$, and the potential energy vanishes.
* the matrix corresponding to ϕ_0, in the sense of exercise 1.4, is $M \equiv \phi_b \mathcal{T}_b = \frac{\vec{x}\cdot\vec{\mathcal{T}}}{r}\, a$.
* spatial derivatives: $[\partial_i, M] = \left(\frac{\mathcal{T}_i}{r} - \frac{x_i \vec{x}\cdot\vec{\mathcal{T}}}{r^3} \right) a$. This does not decrease fast enough to be finite under $\int d^3\vec{x}$.
* but we note, that

$$
\begin{aligned}
[\varepsilon_{ibc} x_b \mathcal{T}_c, x_d \mathcal{T}_d] &= \varepsilon_{ibc} x_b x_d [\mathcal{T}_c, \mathcal{T}_d] \\
&= \varepsilon_{ibc} x_b x_d \, 2i \varepsilon_{cde} \mathcal{T}_e \\
&= 2i\, x_b x_d (\delta_{di}\delta_{be} - \delta_{ei}\delta_{db}) \mathcal{T}_e \\
&= 2i \left(x_i \vec{x}\cdot\vec{\mathcal{T}} - \mathcal{T}_i r^2 \right)
\end{aligned}
$$

* therefore, $[-igA_i, M] = -\left(\frac{\mathcal{T}_i}{r} - \frac{x_i \vec{x}\cdot\vec{\mathcal{T}}}{r^3} \right) a$, if

$$
gA_i = \frac{1}{2r^2} \varepsilon_{ibc} x_b \mathcal{T}_c \, .
$$

This cancels the other term, so that $[D_i, M] \overset{!}{=} 0$.

Winding and flux: Like for a vortex, it can now be shown that

* there is a "winding number", i.e. an integral over fields, which gives a number $n \in \mathbb{Z}$, with $n \neq 0$ if the wrapping is non-trivial.
* for non-trivial wrappings, the monopole sources a magnetic flux, associated with $H \cong SO(2) \cong U(1)$.

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