2. Path integral

2.1 Review of basics

Outline: A derivation of the quantum mechanical (Feynman) path integral has been presented at the end of the lecture "Mechanics II". Here we review some of the key ingredients, changing however the notation towards what is needed in field theory.

Setup: Our "coordinate" is $\hat{\phi}$ and the corresponding momentum is $\hat{\chi}$, with the Hamiltonian

$$\hat{H} = \frac{\hat{\chi}^2}{2m} + V(\hat{\phi}),$$

and the canonical commutator $[\hat{\phi}, \hat{\chi}] = i\hbar$.
Eigenstates satisfy $\hat{\phi}|\phi_i\rangle = \phi_i|\phi_i\rangle$, $\hat{\chi}|\chi_i\rangle = \chi_i|\chi_i\rangle$,
unit operators can be written as

$$\mathbb{1} = \int d\phi\, |\phi\rangle\langle\phi| = \int \frac{d\chi}{2\pi\hbar} |\chi\rangle\langle\chi| ,$$

and the transition amplitude reads $\langle\phi|\chi\rangle = e^{i\chi\phi/\hbar}$.
The time evolution operator is

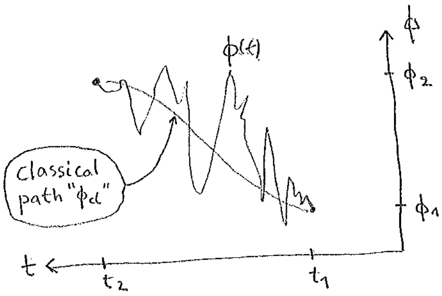
$$\hat{U}(t_2;t_1) = \exp\left(-\frac{i}{\hbar}\hat{H}(t_2-t_1)\right).$$

Formula: According to Feynman,

$$K(\phi_2,t_2;\phi_1,t_1) \equiv \langle\phi_2|\hat{U}(t_2;t_1)|\phi_1\rangle = \int_{\phi(t_1)=\phi_1}^{\phi(t_2)=\phi_2} \mathcal{D}\phi(t)\, \exp\left(\frac{i}{\hbar}S_M\right),$$

$$S_M \equiv \int_{t_1}^{t_2} dt\, \mathcal{L}_M(\phi,\dot{\phi}), \qquad \text{(Minkowskian action)}$$

$$\mathcal{L}_M(\phi,\dot{\phi}) \equiv \frac{m}{2}\dot{\phi}^2 - V(\phi). \qquad \text{(Minkowskian Lagrangian)}$$



Normalization: The integration measure $\mathcal{D}\phi(t)$ is ill-defined, if the time is really a continuous coordinate. A possible way to circumvent this issue is to compute K explicitly in the free limit $V \to 0$:

$$K(\phi_2,t_2;\phi_1,t_1) = \langle\phi_2| e^{-\frac{i}{\hbar}\frac{\hat{\chi}^2}{2m}(t_2-t_1)} |\phi_1\rangle \qquad \boxed{\mathbb{1} = \int \frac{d\chi}{2\pi\hbar}|\chi\rangle\langle\chi|}$$

$$= \int_{-\infty}^{\infty} \frac{d\chi}{2\pi\hbar}\, e^{-\frac{i}{\hbar}\frac{\chi^2(t_2-t_1)}{2m} + \frac{i\chi(\phi_2-\phi_1)}{\hbar}}$$

$$\boxed{\int_{-\infty}^{\infty} dz\, e^{-az^2 \pm ibz} = \sqrt{\frac{\pi}{a}}\, e^{-\frac{b^2}{4a}}}$$

$$= \frac{1}{2\pi\hbar}\sqrt{\frac{\pi\,\hbar 2m}{i(t_2-t_1)}}\; e^{-\frac{(\phi_2-\phi_1)^2\cdot \hbar 2m}{\hbar^2\cdot 4\cdot i(t_2-t_1)}}$$

$$= \sqrt{\frac{m}{2\pi i\hbar(t_2-t_1)}}\; e^{+\frac{im(\phi_2-\phi_1)^2}{2\hbar(t_2-t_1)}}.$$

<u>Harmonic oscillator:</u> Let us apply the path integral to a test case, harmonic oscillator (HO), with $V(\phi) \equiv \frac{1}{2} m\omega^2 \phi^2$.

<u>Semiclassical expansion:</u> We write a general "path" around a classical solution ϕ_{cl} as

$$\phi(t) = \phi_{cl}(t) + \delta\phi(t) \quad ; \quad \phi_{cl}(t_i) = \phi_i \ ; \quad \delta\phi(t_i) = 0 .$$

Then the action becomes

$$S_M = \frac{m}{2} \int_{t_1}^{t_2} dt \left[\dot\phi(t)\dot\phi(t) - \omega^2 \phi(t)\phi(t) \right]$$

$\phi = \phi_{cl} + \delta\phi$; $S_M[\phi_{cl}] \equiv S_{cl}$

$$= S_{cl} + m\int_{t_1}^{t_2} dt \left[\dot\phi_{cl}\,\delta\dot\phi - \omega^2 \phi_{cl}\delta\phi \right] + \frac{m}{2}\int_{t_1}^{t_2} dt \left[\delta\dot\phi\,\delta\dot\phi - \omega^2 \delta\phi\,\delta\phi \right]$$

partial integration and $\delta\phi(t_i) = 0$

$$= S_{cl} + m\int_{t_1}^{t_2} dt\, \underbrace{\left[-\ddot\phi_{cl} - \omega^2\phi_{cl} \right]}_{0!} \delta\phi + \frac{m}{2}\int_{t_1}^{t_2} dt\, \delta\phi \underbrace{\left[-\frac{d^2}{dt^2} - \omega^2 \right]}_{\neq 0} \delta\phi .$$

classical equation of motion

no classical path!

<u>Classical action:</u> Write

$$\phi_{cl}(t) = A\cos\omega(t-t_1) + B\sin\omega(t-t_1)$$

$$\phi_{cl}(t_1) = \phi_1 \quad \Rightarrow \quad A = \phi_1$$

$$\phi_{cl}(t_2) = \phi_2 \quad \Rightarrow \quad \phi_1\cos\omega(t_2-t_1) + B\sin\omega(t_2-t_1) = \phi_2$$

$$\Rightarrow \quad B = \frac{\phi_2 - \phi_1\cos\omega(t_2-t_1)}{\sin\omega(t_2-t_1)} .$$

Then

$$S_{cl} = \frac{m}{2}\int_{t_1}^{t_2} dt \left[\dot\phi_{cl}\dot\phi_{cl} - \omega^2 \phi_{cl}\phi_{cl} \right] \overset{\text{partial integration}}{=} \frac{m}{2}\left[\phi_{cl}\dot\phi_{cl} \right]_{t_1}^{t_2}$$

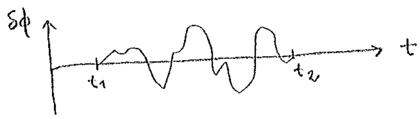
$$= \frac{m\omega}{2}\left\{ \left[\phi_1\cos\omega(t-t_1) + \frac{\phi_2 - \phi_1\cos\omega(t_2-t_1)}{\sin\omega(t_2-t_1)}\sin\omega(t-t_1) \right] \times \left[-\phi_1\sin\omega(t-t_1) + \frac{\phi_2 - \phi_1\cos\omega(t_2-t_1)}{\sin\omega(t_2-t_1)}\cos\omega(t-t_1) \right] \right\}_{t_1}^{t_2}$$

$C \equiv \cos\omega(t_2-t_1)$
$S \equiv \sin\omega(t_2-t_1)$

$$= \frac{m\omega}{2\sin\omega(t_2-t_1)} \left\{ \left[\cancel{\phi_1 C} + \phi_2 - \cancel{\phi_1 C} \right]\left[-\phi_1 S^2 + \phi_2 C - \phi_1 C^2 \right] - \phi_1\left[\phi_2 - \phi_1 C \right] \right\}$$

$$= \frac{m\omega}{2\sin\omega(t_2-t_1)} \left\{ (\phi_2^2 + \phi_1^2)\cos\omega(t_2-t_1) - 2\phi_1\phi_2 \right\} .$$

<u>Quantum fluctuations:</u> Let us write $\delta\phi$ in Fourier representation:



$$\delta\phi = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi(t-t_1)}{t_2-t_1} \right)$$

$$\Rightarrow \frac{m}{2}\int_{t_1}^{t_2} dt\, \delta\phi\left[-\frac{d^2}{dt^2} - \omega^2 \right]\delta\phi = \frac{m}{2}\sum_{n,n'=1}^{\infty} a_n a_{n'} \int_{t_1}^{t_2} dt\, \sin(\ldots)\left(\left(\frac{n\pi}{t_2-t_1}\right)^2 - \omega^2 \right)\sin(\ldots)$$

$$= \frac{m(t_2-t_1)}{4}\sum_{n=1}^{\infty} a_n^2 \left(\left(\frac{n\pi}{t_2-t_1}\right)^2 - \omega^2 \right) .$$

$\langle \sin^2 \rangle_{\text{period}} = \frac{1}{2}$

→

<u>Path integral:</u> We can now insert the classical action and the fluctuation part into the path integral from p. 33. The integration measure is written as

$$\mathcal{D}\phi(t) = J \prod_{n=1}^{\infty} da_n ,$$

where the Jacobian J is to be fixed later on.

$$\Rightarrow K(\phi_2,t_2;\phi_1,t_1) = e^{\frac{i}{\hbar}S_{cl}} \, J \int_{-\infty}^{\infty} \prod_{n=1}^{\infty} da_n \exp\left\{ \frac{im(t_2-t_1)}{4\hbar} \sum_n a_n^2 \left[\left(\frac{n\pi}{t_2-t_1}\right)^2 - \omega^2 \right] \right\}$$

$$= e^{\frac{i}{\hbar}S_{cl}} \, J \prod_{n=1}^{\infty} \left\{ \sqrt{\frac{4\pi\hbar}{im(t_2-t_1)}} \left[\left(\frac{n\pi}{t_2-t_1}\right)^2 - \omega^2 \right]^{-\frac{1}{2}} \right\} .$$

<u>Normalization:</u> The Jacobian J is independent of the potential V, and therefore independent of ω. So we can deduce it from the free limit on p. 33, corresponding to $\omega \to 0$:

$$J \prod_{n=1}^{\infty} \left\{ \sqrt{\frac{4\pi\hbar}{im(t_2-t_1)}} \left[\left(\frac{n\pi}{t_2-t_1}\right)^2 - \omega^2 \right]^{-\frac{1}{2}} \right\}$$

$$= \underbrace{J \prod_{n=1}^{\infty} \left\{ \sqrt{\frac{4\pi\hbar}{im(t_2-t_1)}} \left[\left(\frac{n\pi}{t_2-t_1}\right)^2 \right]^{-\frac{1}{2}} \right\}}_{\text{p.33: } \sqrt{\frac{m}{2\pi i\hbar(t_2-t_1)}}} \times \underbrace{\prod_{n=1}^{\infty} \left\{ \left[1 - \left(\frac{\omega(t_2-t_1)}{n\pi}\right)^2 \right]^{-\frac{1}{2}} \right\}}_{\left\{ \frac{\sin\omega(t_2-t_1)}{\omega(t_2-t_1)} \right\}^{-\frac{1}{2}}}$$

[Arfken (5.210); valid within a radius of convergence.]

<u>Summary:</u>

$$K(\phi_2,t_2;\phi_1,t_1) = \underbrace{\sqrt{\frac{m\omega}{2\pi i\hbar \sin\omega(t_2-t_1)}}}_{\boxed{\text{quantum fluctuations as a prefactor}}} \exp \underbrace{\left\{ \frac{im\omega}{2\sin\omega(t_2-t_1)} \left[(\phi_1^2+\phi_2^2)\cos\omega(t_2-t_1) - 2\phi_1\phi_2 \right] \right\}}_{\boxed{\frac{iS_{cl}}{\hbar}}} .$$

This is the so-called Mehler's formula. From it, the energy eigenvalues (exercise 9.3) and the corresponding wave functions (exercise 10.1) can be deduced.

Green's functions: The object $K(\phi_2,t_2;\phi_1,t_1)$ is an example of a "Green's function", representing the solution of a partial differential equation with specific boundary and/or initial conditions (cf. exercise 9.1). In terms of energy eigenstates $|n\rangle$, it can be expressed as

$$K(\phi_2,t_2;\phi_1,t_1) = \langle\phi_2| e^{-\frac{i}{\hbar}\hat{H}(t_2-t_1)} |\phi_1\rangle \qquad \mathbb{1} = \sum_{n=0}^{\infty} |n\rangle\langle n|$$

$$= \sum_{n=0}^{\infty} \langle\phi_2|n\rangle\langle n|\phi_1\rangle \exp\left\{-\frac{i}{\hbar}E_n(t_2-t_1)\right\}.$$

We now define another Green's function, the "time-ordered propagator":

$$G_T(t_2;t_1) \equiv \langle 0|\hat{T}\{\hat{\phi}_H(t_1)\,\hat{\phi}_H(t_2)\}|0\rangle \equiv \begin{cases} \langle 0|\hat{\phi}_H(t_2)\hat{\phi}_H(t_1)|0\rangle, & t_2 > t_1 \\ \langle 0|\hat{\phi}_H(t_1)\hat{\phi}_H(t_2)|0\rangle, & t_1 > t_2 \end{cases}$$

vacuum state

time ordering

Heisenberg operator

Letting $t_2 > t_1$,

$$G_T(t_2;t_1) = \langle 0|\underbrace{\overbrace{e^{\frac{i}{\hbar}\hat{H}t_2}}^{\hat{U}(0;t_2)}}_{E_0}\hat{\phi}\,\overbrace{e^{-\frac{i}{\hbar}\hat{H}(t_2-t_1)}}^{\hat{U}(t_2;t_1)}\,\hat{\phi}\,\underbrace{\overbrace{e^{-\frac{i}{\hbar}\hat{H}t_1}|0\rangle}^{\hat{U}(t_1;0)}}_{E_0}$$

$$\mathbb{1} = \sum_n |n\rangle\langle n|$$

$$= \sum_{n=0}^{\infty} |\langle 0|\hat{\phi}|n\rangle|^2 \exp\left\{-\frac{i}{\hbar}(E_n-E_0)(t_2-t_1)\right\}.$$

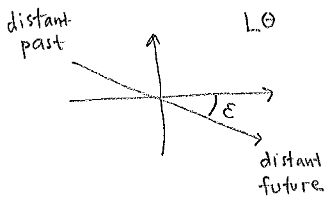
Consider also a third correlator, a mixture of K and G_T:

$$\hat{K} \equiv \langle\phi_2|e^{-\frac{i}{\hbar}\hat{H}\Theta}\,\hat{\phi}_H(t_2)\,\hat{\phi}_H(t_1)\,e^{-\frac{i}{\hbar}\hat{H}\Theta}|\phi_1\rangle \qquad (t_2 > t_1)$$

$$= \langle\phi_2|\underbrace{e^{-\frac{i}{\hbar}\hat{H}(\Theta-t_2)}}_{\hat{U}(0;t_2)}\,\hat{\phi}\,\underbrace{e^{-\frac{i}{\hbar}\hat{H}(t_2-t_1)}}_{\hat{U}(t_2;t_1)}\,\hat{\phi}\,\underbrace{e^{-\frac{i}{\hbar}\hat{H}(t_1+\Theta)}}_{\hat{U}(t_1;-\Theta)}|\phi_1\rangle$$

$$\mathbb{1} = \sum_p |p\rangle\langle p| \qquad \mathbb{1} = \sum_n |n\rangle\langle n| \qquad \mathbb{1} = \sum_q |q\rangle\langle q|$$

$$= \sum_{n,p,q} \langle\phi_2|p\rangle\langle p|\hat{\phi}|n\rangle\langle n|\hat{\phi}|q\rangle\langle q|\phi_1\rangle\, e^{-\frac{i}{\hbar}(E_p+E_q)\Theta - \frac{i}{\hbar}(E_n-E_p)t_2 + \frac{i}{\hbar}(E_n-E_q)t_1}.$$



Suppose that we then give Θ a small imaginary part ("Wick rotation"): $\Theta \to \tilde{\Theta} = \Theta e^{-i\varepsilon}$. Sending $\Theta \to \infty$, all states above the ground state ($p = q = 0$) are exponentially suppressed, and

$$\hat{K} \approx \sum_n \langle\phi_2|0\rangle\, |\langle 0|\hat{\phi}|n\rangle|^2\, \langle 0|\phi_1\rangle\, e^{-\frac{i}{\hbar}(E_n-E_0)(t_2-t_1)}\, e^{-\frac{2iE_0\tilde{\Theta}}{\hbar}}.$$

Here we recognize G_T. The extra factors correspond to

$$\langle\phi_2|\,e^{-\frac{2i\hat{H}\tilde{\Theta}}{\hbar}}\,|\phi_1\rangle = \langle\phi_2|\hat{U}(\Theta;-\Theta)|\phi_1\rangle.$$

So, in the end,

$$G_T = \lim_{\Theta\to\infty} \frac{\langle\phi_2|\hat{T}\{\hat{U}(\tilde{\Theta};t_2)\,\hat{\phi}\,\hat{U}(t_2;t_1)\,\hat{\phi}\,\hat{U}(t_1,-\tilde{\Theta})\}|\phi_1\rangle}{\langle\phi_2|\hat{U}(\tilde{\Theta};-\tilde{\Theta})|\phi_1\rangle}.$$

Both terms on the right can be written as path integrals, but it is a <u>ratio</u>, so the ill-defined measure cancels out!