1.8 Global group properties

Outline:

So far we assumed that the characteristics of a group could be deduced from "small transformations", i.e. group elements close to the unit element, parametrized by the Lie algebra. However, sometimes this does not reveal the full picture, and a more global analysis is required.

Definitions:

Let S be a set, and perm(S) the set of invertible transformations of S onto itself ("permutations").

Note: this is a generalization of linear representations, where $S \to V$ and perm$(S) \to GL(V)$.

We say that the group G acts on S from the left, if there is a homomorphism $L: G \to \mathrm{perm}(S)$, $g \mapsto L_g$:

$$(L_{g_2} \cdot L_{g_1})(x) \equiv L_{g_2}(L_{g_1}(x)) \underset{\text{homomorphism}}{=} L_{g_2 \cdot g_1}(x) .$$

We adopt the useful notation

$$L_g(x) \equiv gx .$$

Similarly a right group action satisfies

$$(R_{g_2} \cdot R_{g_1})(x) \equiv R_{g_2}(R_{g_1}(x)) = R_{g_1 \cdot g_2}(x) ,$$

and is denoted by $R_g(x) \equiv xg$.

The orbit of x under G is the subset

$$O_x \equiv \{ L_g(x) \mid g \in G \} .$$

The point x defines a little group, namely those elements of G which do not move x:

$$G_x \equiv \{ g \in G \mid L_g(x) = x \} .$$

Every little group is a subgroup of G:

(i) $h_1, h_2 \in G_x \implies L_{h_1 \cdot h_2}(x) = L_{h_1}(L_{h_2}(x)) = L_{h_1}(x) = x$

$\implies h_1 \cdot h_2 \in G_x$

(ii) First we note that L_e is an identity mapping:

$$L_g(x) = L_{e \cdot g}(x) = L_e(L_g(x)) \quad \forall g, x .$$

It follows that

$$h \in G_x \implies x = L_e(x) = L_{h^{-1} \cdot h}(x) = L_{h^{-1}}(\underbrace{L_h(x)}_{x})$$

$$\implies h^{-1} \in G_x .$$

Equivalence classes: We define an equivalence relation "$\sim$" within a set so that

(i) $a \sim a$ (reflexivity)

(ii) $a \sim b \iff b \sim a$ (symmetry)

(iii) $a \sim b$ and $b \sim c \Rightarrow a \sim c$ (transitivity).

An equivalence class is a set of all elements equivalent to each other.

The orbits O_x (p. 29) partition S into distinct equivalence classes!

(i) L_e is identity mapping (cf. p. 29) $\Rightarrow x \in O_x$.

(ii) $y \in O_x \Rightarrow y = L_g(x) \Rightarrow x = L_{g^{-1}}(y) \Rightarrow x \in O_y$.

(iii) $y \in O_x$ and $z \in O_y \Rightarrow z = L_g(y) = L_g(L_{g'}(x)) = L_{gg'}(x)$

$\Rightarrow z \in O_x$.

Group acting on itself: We can define left and right actions of G on G, e.g.

$$L_g(g') \equiv g \cdot g' .$$

In this case $O_{g'} = G$, given that an arbitrary element $g'' \in G$ can be reached from g' with $g'' \cdot g'^{-1}$. Such an action is called "transitive".

Another action of G on itself is called conjugation, $g' \mapsto g g' g^{-1}$. In particular a conjugacy class is defined as

$$Cl(g') \equiv \{ g g' g^{-1} \mid g \in G \}.$$

This action cannot be transitive, since $Cl(e) = \{e\}$.

We can also define left and right actions of a subgroup H on G,

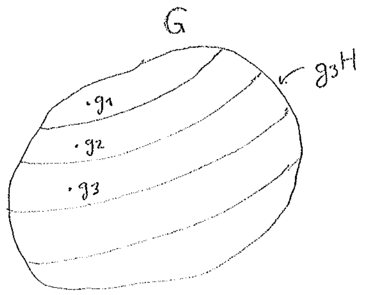
$$L_h(g') \equiv h \cdot g' \quad \text{and} \quad R_h(g') \equiv g' \cdot h .$$

These actions are normally not transitive.

Cosets: The orbit obtained with the right action of H on g,

$$gH \equiv \{ g \cdot h \mid h \in H \} ,$$

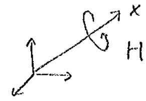
is called a left coset of H in G. Similarly, we can define right cosets.



Varying g, and multiplying each with the full H, we obtain a splitting of G into disjoint cosets, each of which represents an equivalence class. The set of left cosets is denoted by

$$G/H \qquad \text{"G modulo H"} .$$

Example: Let H be the little group of $x \in S$, i.e. the elements $h \in H \subset G$ which do not transform x. If we illustrate the situation with rotations, then H corresponds to rotations along the direction of x:



H = "rotations around x-direction"

The set G/H then represents the classes of rotations that do change x. So, the set G/H can be parametrized by the directions of x.

Cosets as group elements? The set G/H consists of a collection of sets (left cosets). We would now like to ask under which conditions we can define a group multiplication between the cosets?

Attempt: $(g_1 H) \cdot (g_2 H) \overset{?}{=} g_1 g_2 H$.

Problem: $g_1 H = g_1 h H \quad \forall\, h \in H$.

So, if the attempt were correct, we should have

$$g_1 h g_2 H \overset{?}{=} g_1 g_2 H \qquad \forall\, h \in H .$$

This would require that $\exists\, h' \in H$ with

$$g_1 h\, g_2 = g_1 g_2 h'$$

$\Leftrightarrow \forall h \in H, \forall g_2 \in G \;\; \exists\, h' \in H$ so that $h g_2 = g_2 h'$.

This is not necessarily the case.

However, if we choose a subgroup with special properties, the condition can be satisfied!

Normal subgroup: We define a normal subgroup as one which remains invariant in all conjugations:

$$\forall g: \quad g H g^{-1} = \{ g h g^{-1} \mid h \in H \} = H .$$

This implies that

$$\forall g \in G, \forall h \in H \quad \exists\, h' \in H \quad \text{so that} \quad g h g^{-1} = h'$$
$$\Leftrightarrow \quad g h = h' g .$$

Renaming $g \to g_2$, $h \leftrightarrow h'$ we find $h g_2 = g_2 h'$, i.e. the condition above is satisfied.

In other words, for a normal subgroup, G/H has indeed a group structure. We call it a quotient group.

Examples: (i) the <u>center</u> $\mathcal{Z} \equiv \{ z \in G \mid zg = gz \ \forall\, g \in G \}$, i.e. group elements that commute with all others. Clearly $\mathcal{Z}$ is a normal subgroup.

(ii) let f be a homomorphism, and let us define its <u>kernel</u> as

$$\ker(f) \equiv \{ h \in G \mid f(h) = e \}.$$

Then $f(ghg^{-1}) = f(g)\underbrace{f(h)}_{e}f(g^{-1}) = f(g)f(g^{-1}) = f(gg^{-1}) = e$.

So, $ghg^{-1} \in \ker(f)$, and thus $\ker(f)$ is a normal subgroup.

Relation between homomorphisms and isomorphisms

We inspect a homomorphism $f: G \to G'$. Its <u>image</u> is defined as $\mathrm{im}(f) \equiv \{ f(g) \mid g \in G \}$.

Claim: $\underbrace{G/\ker(f)}_{\text{quotient group}} \cong \mathrm{im}(f)$. isomorphism

In other words, a homomorphism induces an isomorphism!

Proof: Let $\varphi : G/\ker(f) \to \mathrm{im}(\varphi)$, with $\varphi(g\underbrace{\ker(f)}_{\equiv H}) \equiv f(g)$.

(i) φ is a homomorphism:

$$\varphi(g_1 H g_2 H) = \varphi(g_1 g_2 H) = f(g_1 g_2) = f(g_1)f(g_2) = \varphi(g_1 H)\varphi(g_2 H).$$

normal subgroup

(ii) φ is a surjective by definition [$\mathrm{im}(f)$].

(iii) φ is injective:

$$\varphi(g_1 H) = \varphi(g_2 H) \iff f(g_1) = f(g_2)$$
$$\Rightarrow e = [f(g_1)]^{-1} f(g_2) = f(g_1^{-1}) f(g_2) = f(g_1^{-1} g_2)$$
$$\Rightarrow g_1^{-1} g_2 = h \in \ker(f) \Rightarrow g_2 = g_1 h$$
$$\Rightarrow g_2 H = g_1 H. \quad \square$$

Summary: If two groups are homomorphic, they have the same Lie algebras, weight diagrams, reduction rules, etc. However, they are not necessarily isomorphic. To know if this is the case, some of their global properties, like the center, need also to be studied. What this implies is illustrated in exercise sheet 8.