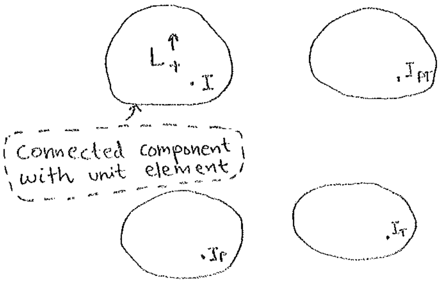
1.7 Lorentz group

Outline:

So far we have illustrated the general formalism with the examples of SU(2) and SU(3). Now we move on to the Lorentz group $L \equiv O(3,1)$, satisfying $A^T \eta A = \eta$, $\eta = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$ (p.3). Specifically, defining

$$\underbrace{I \equiv \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}}_{\text{unit element}}, \quad \underbrace{I_P \equiv \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}}_{\text{spatial reflection}}, \quad \underbrace{I_T \equiv \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}}_{\text{time reversal}}, \quad \underbrace{I_{PT} \equiv \begin{pmatrix} -1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}}_{\text{spacetime reflection}},$$



we are concerned with the group $L_+^\uparrow$ of "proper" Lorentz transformations, which form a Lie group. We would like to classify its irreducible representations.

SL(2, $\mathbb{C}$):

We first show that $L_+^\uparrow$ is homomorphic to $SL(2,\mathbb{C})$ (cf. p.1). This goes in several steps:

(a) We map vectors $x^\mu \in \mathbb{R}^4$ onto Hermitean 2x2-matrices, by making use of the Pauli matrices from p.4:

$$\sigma_0 \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \sigma_1 \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 \equiv \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

It is also convenient to define $\bar{\sigma}^\mu \equiv \sigma_\mu$. Then

$$M \equiv x^\mu \sigma_\mu = \begin{pmatrix} x^0 + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & x^0 - x^3 \end{pmatrix}$$

is Hermitean, and conversely, every Hermitean 2x2 matrix can be written in this form.

The components x^μ can be projected out as

$$x^\mu = \frac{1}{2} \operatorname{tr}(\bar{\sigma}^\mu M).$$

(b) We note that

$$\det M = (x^0)^2 - (x^3)^2 - (x^1)^2 - (x^2)^2$$

$$= -\eta_{\mu\nu} x^\mu x^\nu.$$

we had defined $\eta = \operatorname{diag}(-1,1,1,1)$ on p.3

(c) Let now $A \in SL(2, \mathbb{C})$, and

$$M' \equiv A M A^+.$$

Clearly $M'^+ = A M^+ A^+ = A M A^+ = M'$ remains Hermitean. The transition is linear:

$$(a_1 M_1 + a_2 M_2)' = a_1 M_1' + a_2 M_2'.$$

So we can write $x'^\mu = \frac{1}{2} \operatorname{tr}(\bar{\sigma}^\mu M') = [\Lambda(A)]^\mu{}_\nu \, x^\nu$.

(d) We can now verify that $\Lambda(A)$ is a Lorentz transformation:

$$-\eta_{\mu\nu}x'^{\mu}x'^{\nu} = \det M' = \det AMA^{\dagger}$$
$$= \underbrace{|\det A|^2}_{1} \det M = -\eta_{\mu\nu}x^{\mu}x^{\nu} .$$

(e) Furthermore, $SL(2,\mathbb{C})$ is connected, and $\mathbb{1}_{2\times 2} \in SL(2,\mathbb{C})$ maps to $I \in L_+^{\uparrow}$, so $\Lambda(A) \in L_+^{\uparrow}$.

(f) Finally, we can verify that the mapping $f: A \to \Lambda(A)$ is indeed a homomorphism. For this, consider what $A_1 \cdot A_2$ does to $x^{\mu}\sigma_{\mu}$:

$$A_1 \cdot A_2 \, (x^{\mu}\sigma_{\mu}) \, [A_1 A_2]^{\dagger} = \underbrace{A_1 \underbrace{A_2 x^{\mu}\sigma_{\mu} A_2^{\dagger}}_{[\Lambda(A_2)]^{\alpha}{}_{\mu} x^{\mu}\sigma_{\alpha}} A_1^{\dagger}}_{[\Lambda(A_1)]^{\beta}{}_{\alpha} [\Lambda(A_2)]^{\alpha}{}_{\mu} x^{\mu}\sigma_{\beta}}$$

So indeed $\Lambda(A_1 \cdot A_2) = \Lambda(A_1) \cdot \Lambda(A_2) \Rightarrow \square$.

Remark: We have shown that every $A \in SL(2,\mathbb{C})$ generates a Lorentz transformation in $L_+^{\uparrow}$. We have not shown whether this mapping is a bijection, i.e. whether all elements of $L_+^{\uparrow}$ are uniquely reached. This will be addressed in sec. 1.8.

Bijection would guarantee that the relation is an isomorphism (p.1), but homomorphism already suffices for classifying the irreducible representations, though with caveats.

Summary: We need to work out the Lie algebra generators of

$$SL(2,\mathbb{C}) = \{ A \in GL(2,\mathbb{C}) \mid \det A = 1 \},$$

of which there are $\dim SL(2,\mathbb{C}) = 2(n^2-1)\big|_{n=2} = 6$ (cf. p.2).

Generators:

The goal is to write $A \equiv \exp(i\theta^a T^a)$, $a = 1, \dots, 6$; and then inspect the Cartan subalgebra.

There is, however, an issue: $SL(2,\mathbb{C})$ is not compact
$\Rightarrow$ representations are not necessarily unitary (p. 11)
$\Rightarrow$ generators are not necessarily Hermitean.
This will become apparent in a moment, however the general procedure stays the same.

Now: $\det A = 1 = \exp(i\theta^a \operatorname{tr}[T^a]) \Rightarrow \boxed{\operatorname{tr}(T^a) = 0.}$

A general traceless complex 2×2-matrix has the form

$$\begin{pmatrix} a & b \\ c & -a \end{pmatrix} = \operatorname{Re} a\, \sigma_3 + i \operatorname{Im} a\, \sigma_3 + \frac{b+c}{2}\sigma_1 + \frac{b-c}{2} i\sigma_2$$

$$= \operatorname{Re} a\, \sigma_3 + \frac{\operatorname{Re}(b+c)}{2}\sigma_1 - \frac{\operatorname{Im}(b-c)}{2}\sigma_2$$

$$+ \operatorname{Im} a\, i\sigma_3 + \frac{\operatorname{Im}(b+c)}{2} i\sigma_1 + \frac{\operatorname{Re}(b-c)}{2} i\sigma_2$$

$\Rightarrow$ there are indeed six real parameters!

We define the generators as

$$J_{23} \equiv \tfrac{1}{2}\sigma_1 \qquad J_{10} \equiv -\tfrac{i}{2}\sigma_1$$

$$J_{31} \equiv \tfrac{1}{2}\sigma_2 \qquad J_{20} \equiv -\tfrac{i}{2}\sigma_2$$

$$J_{12} \equiv \tfrac{1}{2}\sigma_3 \qquad J_{30} \equiv -\tfrac{i}{2}\sigma_3 ,$$

together with the rule $J_{\nu\mu} \equiv -J_{\mu\nu}$.

The generators J_{12}, J_{23}, J_{31} generate rotations and are Hermitean, whereas J_{10}, J_{20}, J_{30} generate boosts and are not Hermitean.

The parameters are also renamed, and conventionally we write

$$A = \exp\left(-\tfrac{i}{2}\omega_{\mu\nu} J^{\mu\nu}\right), \quad \omega_{\mu\nu} \in \mathbb{R}.$$

It can be verified that the generators satisfy the Lorentz algebra (cf. exercise 7.2)

$$[J_{\mu\nu}, J_{\rho\sigma}] = -i\left(\eta_{\nu\rho} J_{\mu\sigma} - \eta_{\mu\rho} J_{\nu\sigma} - \eta_{\nu\sigma} J_{\mu\rho} + \eta_{\mu\sigma} J_{\nu\rho}\right).$$

in text books one often sees the opposite sign here, because $\eta = \operatorname{diag}(1,-1,-1,-1)$ there.

Weight diagrams

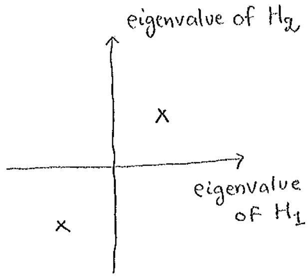
We have seen that the group of generators is composed of two copies of the generators of $SU(2)$.* In particular, two generators can be diagonalized simultaneously, J_{12} and J_{30} $\Rightarrow$ rank = 2.

* In other words, $SL(2,\mathbb{C}) \sim SU(2) \otimes SU(2)$.

From here we can immediately deduce that the irreps can be classified as $\mathcal{D}(s_1, s_2)$, $s_i = 0, \frac{1}{2}, 1, \dots,$ and that $\dim \mathcal{D}(s_1,s_2) = (2s_1+1)(2s_2+1)$.

Cartan subalgebra: $H_1 \equiv J_{12} = \begin{pmatrix} \frac{1}{2} & \\ & -\frac{1}{2} \end{pmatrix}$, $H_2 \equiv iJ_{30} = \begin{pmatrix} \frac{1}{2} & \\ & -\frac{1}{2} \end{pmatrix}$.

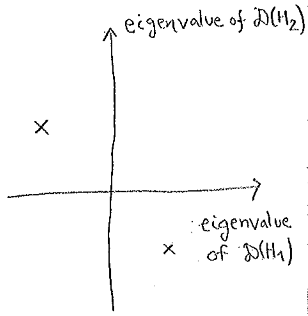
* $D(\frac{1}{2},0)$: eigenvectors $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$



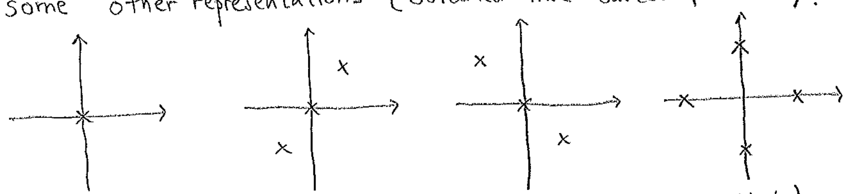
* $D(0,\frac{1}{2})$: to make it independent, we "conjugate" H_2:

$$\mathcal{D}(H_1) = \begin{pmatrix} \frac{1}{2} & \\ & -\frac{1}{2} \end{pmatrix}, \quad \mathcal{D}(H_2) \equiv -iJ_{30} = \begin{pmatrix} -\frac{1}{2} & \\ & \frac{1}{2} \end{pmatrix}.$$

Eigenvectors still $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$:



* some other representations (obtained like direct products):



$D(0,0)$ $\quad$ $D(1,0)$ $\quad$ $D(0,1)$ $\quad$ $D(\frac{1}{2},\frac{1}{2})$

Some physics:

$D(0,0)$ = scalar field ; $D(\frac{1}{2},0)$, $D(0,\frac{1}{2})$ = chiral fermions.

$D(\frac{1}{2},0) \otimes D(\frac{1}{2},0) = D(0,0) \oplus D(1,0)$

The invariant $D(0,0)$ is what appears in the Lagrangian!

Explicit construction: $A \in SL(2,\mathbb{C})$: $\psi'_\alpha = A_\alpha{}^\beta \psi_\beta$.

Consider the tensor $\varepsilon^{\alpha\beta}\psi'_\alpha\psi'_\beta = \underbrace{\varepsilon^{\alpha\beta} A_\alpha{}^\gamma A_\beta{}^\delta}_{\text{p.24: } \det A\, \varepsilon^{\gamma\delta}} \psi_\gamma \psi_\delta = \varepsilon^{\gamma\delta}\psi_\gamma\psi_\delta$ = invariant!

But is not $\varepsilon^{\gamma\delta}\psi_\gamma\psi_\delta = \psi_1\psi_2 - \psi_2\psi_1 = 0$??

No, fermions are Grassmann variables!*

* Incidentally, the group coordinates θ^a can also be generalized to Grassmann variables, leading ultimately to the concept of Supersymmetry.