1.6 Reduction

Outline: We can now classify irreducible representations of a Lie group. However a general representation, like the direct product from p.12, is reducible. How can we reduce it to a direct sum of irreducible representations? To achieve this, we need to work out the corresponding weight diagrams!

Direct sum: (p.12)

* $D_1 \oplus D_2(g)(v_1, v_2) \equiv (D_1(g)v_1, 0) + (0, D_2(g)v_2)$.
* basis vectors: $\{(\hat{e}_i, 0)\}$ together with $\{(0, \hat{e}_j)\}$.
* generators: $\left(\begin{array}{c|c} D_1(T^a) & \\ \hline & 0 \end{array}\right)$ together with $\left(\begin{array}{c|c} 0 & \\ \hline & D_2(T^a) \end{array}\right)$.
* $\left(\begin{array}{c|c} D_1(H_i) & \\ \hline & 0 \end{array}\right)\begin{pmatrix} v^j \\ 0 \end{pmatrix} = \omega_i^j \begin{pmatrix} v^j \\ 0 \end{pmatrix}$ if $D_1(H_i)v^j = \omega_i^j v^j$.

  Therefore the weight vectors (i.e. eigenvalues) are those of D_1 and those of D_2. Their number is the dimension of the sum representation, i.e. $d_1 + d_2$.

Direct product: (p. 12)

* $D_1 \otimes D_2(g)(v_1, v_2) \equiv (D_1(g)v_1, D_2(g)v_2)$
* basis vectors: $\{(\hat{e}_i, \hat{e}_j)\}$.
* to determine the generators, let us inspect how a small transformation, with representations $(i=1,2)$

$$D_i(g) = \exp(i\theta^a D_i(T^a)) = \mathbb{1} + i\theta^a D_i(T^a) + \mathcal{O}(\theta^a)^2,$$

acts in the direct product:

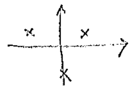
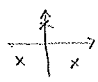
$$D_1 \otimes D_2(g)(v_1, v_2) = (v_1 + i\theta^a D_1(T^a)v_1, v_2 + i\theta^a D_2(T^a)v_2) + \mathcal{O}(\theta^a)^2$$

$$\overset{\text{linear vector space}}{=} (v_1, v_2) + i\theta^a\left[(D_1(T^a)v_1, v_2) + (v_1, D_2(T^a)v_2)\right] + \mathcal{O}(\theta^a)^2.$$

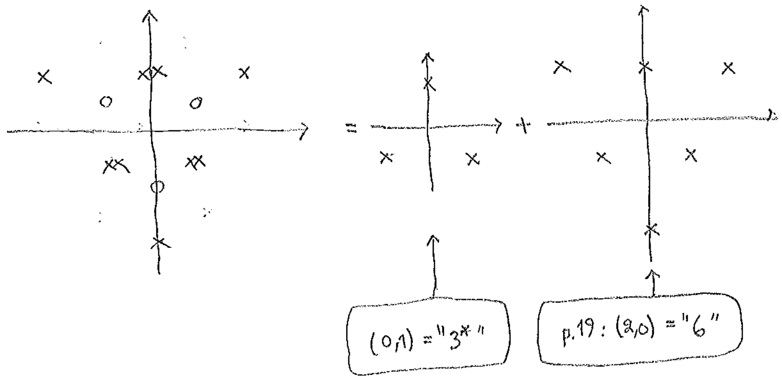
Let us now choose T^a from the Cartan subalgebra, and let the vectors v_1, v_2 be eigenvectors, with $D_1(H_i)v_1 = \omega_i v_1$ and $D_2(H_i)v_2 = \bar{\omega}_i v_2$. Then

$$\left[(D_1(H_i)v_1, v_2) + (v_1, D_2(H_i)v_2)\right] = (\omega_i + \bar{\omega}_i)(v_1, v_2).$$

Therefore the weight vectors (i.e. eigenvalues) are the sums of those of D_1 and D_2. Their number is $d_1 \times d_2$.

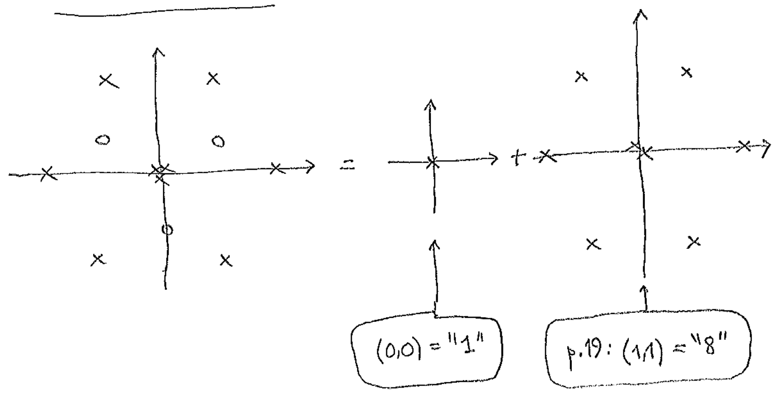
Example: SU(3): From p.19: $(1,0) = "3" =$  $(0,1) = "3^*" =$ 

$\boxed{3 \otimes 3}$: we just sum the diagrams and identify the result as a superposition of irreducible diagrams.



$$\Rightarrow \underline{3 \otimes 3 = 6 \oplus 3^*}\,.$$

$\boxed{3 \otimes 3^*}$



$$\Rightarrow \underline{3 \otimes 3^* = 8 \oplus 1}\,.$$

Example: SU(2): From p.7: there is only one diagonal generator: $H_1 = T^3 = \frac{\sigma_3}{2}$.

The representations are classified by the spin $s = 0, \frac{1}{2}, 1, \ldots$, which in the new general language corresponds to $s = \frac{q^1}{2}$.

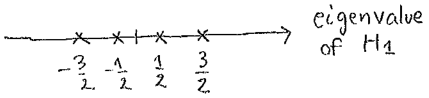
From quantum mechanics, we may already recall the reduction formula for a product representation:

$$"s_1" \otimes "s_2" = "|s_1 - s_2|" \oplus "|s_1 - s_2| + 1" \oplus \cdots \oplus "s_1 + s_2".$$

The dimensions are $d_i = 2s_i + 1 = q_i + 1$.

The weight, i.e. the eigenvalue of H_1, is $\omega = S_z = m$.

All weight vectors of a representation s_i:



$$m_i = -s_i, -s_i + 1, \ldots, s_i - 1, s_i\,.$$

The corresponding eigenvectors are denoted by $v = |s_i\, m_i\rangle$.

Clebsch-Gordan: So far we have considered the reduction of weight vectors, i.e. eigenvalues, but it can also be asked how the eigenvectors get reduced:

coefficients to be determined

$$(v,\tilde{v}) = \sum_{i,j} v_i \tilde{v}_j \underbrace{(\hat{e}_i, \hat{e}_j)}_{\sum_a C^{(1)}_{a;ij}\,\hat{e}^{(1)}_a \;+\; \sum_b C^{(2)}_{b;ij}\,\hat{e}^{(2)}_b \;+\; \dots}$$

basis vectors of irreps

The idea is to start from the eigenstate corresponding to the highest weight w^1, and then operate with $\mathcal{D}(E_{-k})$ on it, with the help of the coefficients $N_{-\alpha,w}$ from p. 20. We illustrate this with $SU(2)$, where the $C^{(m)}_{a;ij}$ are known as the Clebsch-Gordan coefficients.

For $SU(2)$: $\mathcal{D}(E_{-k}) \to \mathcal{D}(S_-)$ (lowering operator)

$$\mathcal{D}(S_\pm)|s\,m\rangle = \sqrt{s(s+1) - m(m\pm 1)}\;|s\;m\pm 1\rangle \,.$$

Furthermore, the eigenvalue of $|s_1 m_1\rangle|s_2 m_2\rangle$ is $m_1 + m_2$, and among "$|s_1 - s_2|$" $\oplus \dots \oplus$ "s_1+s_2", only "s_1+s_2" can have the eigenvalue s_1+s_2. So we deduce

$$|s_1 s_1\rangle|s_2 s_2\rangle = |s_1+s_2\;\; s_1+s_2\rangle\,,$$

and can now operate on both sides with $\mathcal{D}(S_-)$.

Example: How can $|\tfrac{1}{2}\tfrac{1}{2}\rangle|\tfrac{1}{2}\!-\!\tfrac{1}{2}\rangle$ be expressed in terms of $|1m\rangle$ and $|00\rangle$?

Starting point: $|\tfrac{1}{2}\tfrac{1}{2}\rangle|\tfrac{1}{2}\tfrac{1}{2}\rangle = |1\,1\rangle$

Operate on both sides with $\mathcal{D}(S_-)$, using $\sqrt{\tfrac{1}{2}\cdot\tfrac{3}{2} + \tfrac{1}{2}\cdot\tfrac{1}{2}} = 1$, $\sqrt{1\cdot 2 - 1\cdot 0} = \sqrt{2}$

$$\Rightarrow \quad |\tfrac{1}{2}\!-\!\tfrac{1}{2}\rangle|\tfrac{1}{2}\tfrac{1}{2}\rangle + |\tfrac{1}{2}\tfrac{1}{2}\rangle|\tfrac{1}{2}\!-\!\tfrac{1}{2}\rangle = \sqrt{2}\,|1\,0\rangle\,.$$

Then operate again, using $\sqrt{\tfrac{1}{2}\cdot\tfrac{3}{2} - \tfrac{1}{2}\cdot\tfrac{3}{2}} = 0$, $\sqrt{1\cdot 2 + 0\cdot 1} = \sqrt{2}$

$$\Rightarrow \quad 2\,|\tfrac{1}{2}\!-\!\tfrac{1}{2}\rangle|\tfrac{1}{2}\!-\!\tfrac{1}{2}\rangle = 2\,|1\!-\!1\rangle\,.$$

$$\Rightarrow \quad |1\pm 1\rangle = |\tfrac{1}{2}\pm\tfrac{1}{2}\rangle|\tfrac{1}{2}\pm\tfrac{1}{2}\rangle\,,\quad |10\rangle = \tfrac{1}{\sqrt{2}}\big(|\tfrac{1}{2}\!-\!\tfrac{1}{2}\rangle|\tfrac{1}{2}\tfrac{1}{2}\rangle + |\tfrac{1}{2}\tfrac{1}{2}\rangle|\tfrac{1}{2}\!-\!\tfrac{1}{2}\rangle\big).$$

We still need $|00\rangle = \alpha\,|\tfrac{1}{2}\!-\!\tfrac{1}{2}\rangle|\tfrac{1}{2}\tfrac{1}{2}\rangle + \beta\,|\tfrac{1}{2}\tfrac{1}{2}\rangle|\tfrac{1}{2}\!-\!\tfrac{1}{2}\rangle$.

Operate on both sides with $\mathcal{D}(S_-)$

$$\Rightarrow \quad 0 = (\alpha+\beta)\,|\tfrac{1}{2}\!-\!\tfrac{1}{2}\rangle|\tfrac{1}{2}\!-\!\tfrac{1}{2}\rangle \quad \Rightarrow \alpha = -\beta$$

$$\Rightarrow \quad |00\rangle = \tfrac{1}{\sqrt{2}}\big(-|\tfrac{1}{2}\!-\!\tfrac{1}{2}\rangle|\tfrac{1}{2}\tfrac{1}{2}\rangle + |\tfrac{1}{2}\tfrac{1}{2}\rangle|\tfrac{1}{2}\!-\!\tfrac{1}{2}\rangle\big)$$

overall sign is a convention

$$\Rightarrow \quad |\tfrac{1}{2}\tfrac{1}{2}\rangle|\tfrac{1}{2}\!-\!\tfrac{1}{2}\rangle = \tfrac{1}{\sqrt{2}}\big(|10\rangle + |00\rangle\big).$$

Tensor method: Another way to relate vectors makes use of symmetries. Let us consider a direct product of a number of fundamental and conjugated representations. A general vector reads

Sums over repeated indices implied

$$v \equiv \underbrace{v^{i_1 i_2 \cdots i_n}_{j_1 j_2 \cdots j_m}}_{\text{"tensor"}} (\underbrace{\hat{e}_{i_1}, \hat{e}_{i_2}, \dots, \hat{e}_{i_n}}_{\text{fundamental}}, \underbrace{\hat{e}^{j_1}, \hat{e}^{j_2}, \dots, \hat{e}^{j_m}}_{\text{conjugated}})$$

We denote:

$$D(g)\,\hat{e}_{i_1} = g\,\hat{e}_{i_1} \equiv \hat{e}_{\tilde{i}_1} g_{\tilde{i}_1 i_1} \equiv g^{\tilde{i}_1}_{i_1} \hat{e}_{\tilde{i}_1},$$

$$D^*(g)\,\hat{e}^{j_1} = g^*\,\hat{e}^{j_1} \equiv \hat{e}^{\tilde{j}_1} g^*_{\tilde{j}_1 j_1} \equiv g^{*\,j_1}_{\tilde{j}_1} \hat{e}^{\tilde{j}_1}.$$

Then

$$D(g)v = \tilde{v}^{\tilde{i}_1 \tilde{i}_2 \cdots \tilde{i}_n}_{\tilde{j}_1 \tilde{j}_2 \cdots \tilde{j}_m} (\hat{e}_{\tilde{i}_1}, \hat{e}_{\tilde{i}_2}, \dots, \hat{e}_{\tilde{i}_n}, \hat{e}^{\tilde{j}_1}, \hat{e}^{\tilde{j}_2}, \dots, \hat{e}^{\tilde{j}_m}),$$

with

$$\tilde{v}^{\tilde{i}_1 \tilde{i}_2 \cdots \tilde{i}_n}_{\tilde{j}_1 \tilde{j}_2 \cdots \tilde{j}_m} = g^{\tilde{i}_1}_{i_1} g^{\tilde{i}_2}_{i_2} \cdots g^{\tilde{i}_n}_{i_n}\, g^{*\,j_1}_{\tilde{j}_1} g^{*\,j_2}_{\tilde{j}_2} \cdots g^{*\,j_m}_{\tilde{j}_m}\, v^{i_1 i_2 \cdots i_n}_{j_1 j_2 \cdots j_m}.$$

We now note that:

- if v is symmetric in $(i_a \leftrightarrow i_b)$ or $(j_a \leftrightarrow j_b)$, then $\tilde{v}$ is symmetric in $(\tilde{i}_a \leftrightarrow \tilde{i}_b)$ or $(\tilde{j}_a \leftrightarrow \tilde{j}_b)$.
- if v is antisymmetric in $(i_a \leftrightarrow i_b)$ or $(j_a \leftrightarrow j_b)$, then $\tilde{v}$ is antisymmetric in $(\tilde{i}_a \leftrightarrow \tilde{i}_b)$ or $(\tilde{j}_a \leftrightarrow \tilde{j}_b)$.

Therefore, symmetric and antisymmetric tensors build invariant subspaces. This is a necessary (if not sufficient) condition for identifying irreducible representations.

Another invariant subspace is found by noting that

- if v has a "trace part", $\delta^{i_a}_{j_b}$, then $\tilde{v}$ has a trace part $\delta^{\tilde{i}_a}_{\tilde{j}_b}$.

Proof:

$$g^{\tilde{i}_a}_{i_a} g^{*\,j_b}_{\tilde{j}_b} \delta^{i_a}_{j_b} = g^{\tilde{i}_a}_{i_a} g^{*\,i_a}_{\tilde{j}_b} \overset{\text{(above)}}{=} g_{\tilde{i}_a i_a} g^*_{\tilde{j}_b i_a}$$

$$= g_{\tilde{i}_a i_a} g^\dagger_{i_a \tilde{j}_b} = (gg^\dagger)_{\tilde{i}_a \tilde{j}_b} = \delta^{\tilde{i}_a}_{\tilde{j}_b}.$$

if g is unitary

So, trace parts yield further invariant subspaces.

Finally we recall that $\varepsilon_{i_1 i_2 \cdots i_n} g_{i_1 j_1} g_{i_2 j_2} \cdots g_{i_n j_n} = \det g\, \varepsilon_{j_1 j_2 \cdots j_n}$, so if $\det g = 1$, tensors proportional to the Levi-Civita symbol yield further invariant subspaces.

We will see in exercise sheet 6 how all this works!