1.5 Classification of representations

Ingredients:

- $*$ Cartan subalgebra: H_i, $i=1,\dots,m$, $m=$rank, $[H_i, H_j]=0$.
- $*$ d-dimensional representation: $\mathcal{D}(H_i)$ as $d\times d$ - matrices.
- $*$ weights: m-dimensional vectors $\omega^j \in \mathbb{R}^m$, $j=1,\dots,d$.
- $*$ roots: m-dimensional vectors $\alpha^k \in \mathbb{R}^m$, $k=1,\dots,$ dim**.

** m of these are zero vectors, see p.16

- $*$ simple positive roots α^k, $k=1,\dots,m$: a good basis for $\mathbb{R}^m$!
- $*$ matrices $\mathcal{D}(E_k)$, $\mathcal{D}(E_{-k}) \equiv \mathcal{D}(E_k^+)$: bring us from one eigenvector to another, so that the corresponding eigenvalues change as $\omega^j \to \omega^j + \alpha^k$ or $\omega^j \to \omega^j - \alpha^k$.

Highest weight: We had introduced an ordering $\omega^1 > \omega^2 > \dots$ (p.14).
If we operate with a positive root $\mathcal{D}(E_k)$, $k>0$, on the eigenvector corresponding to ω^1, the result must vanish, $\mathcal{D}(E_k)v^1 = 0$. I.e. we can only go down from ω^1.

Construction: The concept of the highest weight, together with a "main formula", derived on p.20, lead to a unique classification of irreducible representations. The main formula asserts that:

$$\forall k,j \; : \quad \frac{2\alpha^k \cdot \omega^j}{|\alpha^k|^2} = q - p \; ;$$

$p = 0,1,2,\dots =$ number of operations with $\mathcal{D}(E_k)$, yielding a non-vanishing vector.

$q = 0,1,2,\dots =$ number of operations with $\mathcal{D}(E_{-k})$, yielding a non-vanishing vector.

For the highest weight ω^1 with simple positive roots α^k: $p=0$.

Therefore ω^1 is fully determined by its projections in the directions of the simple positive roots:

$$q^k \equiv \frac{2\alpha^k \cdot \omega^1}{|\alpha^k|^2} \in \{0,1,2,\dots\}, \quad k=1,\dots,m.$$

The corresponding representation is identified as

$$(q^1, q^2, \dots, q^m).$$

All other weights are reached by operating with $\mathcal{D}(E_{-k})$.

Fundamental weights: In order to be able to draw weight diagrams, it is helpful to first find "fundamental weights", μ^k, defined through

$$\frac{2\alpha^k\cdot\mu^l}{|\alpha^k|^2} = \delta^{kl}, \quad k,l\in\{1,\dots,m\}.$$

Then w^1 of representation $(q^1,q^2,\dots,q^m)$ is just

$$w^1 = \sum_{k=1}^{m} q^k\mu^k.$$

Example: SU(3) p.16: $\alpha^1 = \left(\frac{1}{2},\frac{\sqrt{3}}{2}\right)$, $\alpha^2 = \left(\frac{1}{2},-\frac{\sqrt{3}}{2}\right)$

$$|\alpha^1|^2 = \frac{1+3}{4} = 1, \quad |\alpha^2|^2 = \frac{1+3}{4} = 1.$$

To construct the fundamental weights, we thus look at $2\alpha^k\cdot\mu^l = \delta^{kl}$.

$\boxed{\mu^1 \equiv (a,b)}$

$\alpha^1:\ a+\sqrt{3}\,b = 1$

$\alpha^2:\ a-\sqrt{3}\,b = 0$

$\Rightarrow 2a = 1;\ a = \frac{1}{2};\ b = \frac{1}{2\sqrt{3}}$

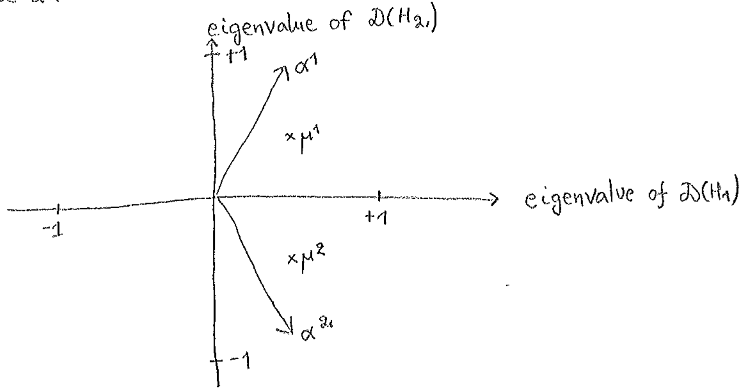
$\boxed{\mu^2 \equiv (c,d)}$

$\alpha^1:\ c+\sqrt{3}\,d = 0$

$\alpha^2:\ c-\sqrt{3}\,d = 1$

$\Rightarrow 2c = 1;\ c = \frac{1}{2};\ d = -\frac{1}{2\sqrt{3}}$

In a principal plot, we include μ^1 and μ^2 [since these determine $w^1 = q^1\mu^1 + q^2\mu^2$ of irrep (q^1,q^2)] as well as α^1 and α^2 [since these allow us to reach the other $w^j = w^1 - n^1\alpha^1 - n^2\alpha^2$, $n^i = 0,1,2,\dots$].



There are a few general properties that are helpful to know:

* all representation have "symmetrical" diagrams
* conjugated representation (cf. exercise 5.4):

$$(q^1,q^2)^* = (q^2,q^1) \ \hat{=}\ \text{reflection across x-axis}$$

* dimension of the representation = number of w^j:

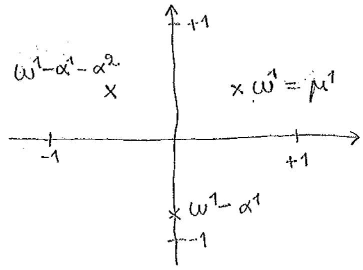
$$d = \frac{(q^1+1)(q^2+1)(q^1+q^2+2)}{2}.$$

Weight diagrams: $*\ (1,0)$; $d=3$

we cannot operate with α^2 on ω^1

$\omega^1 = \mu^1$; we can go down once with α^1 from ω^1

"fundamental" representation 3

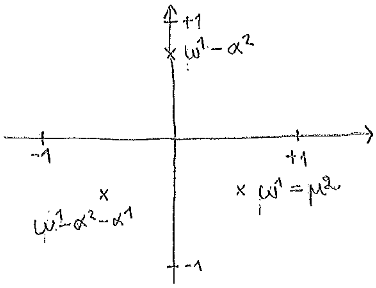


$*\ (0,1)$; $d=3$

$\omega^1 = \mu^2$; we can go down once with α^2 from ω^1

we cannot operate with α^1 on ω^1

"conjugated" or "antifundamental" representation 3^*

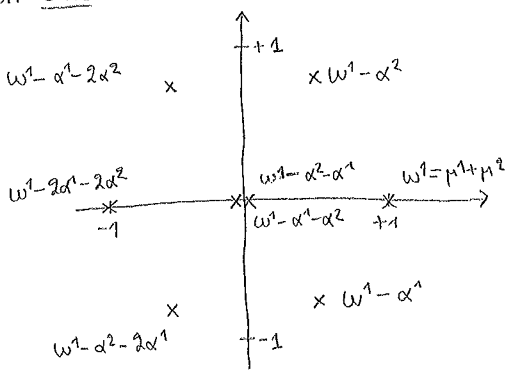


$*\ (1,1)$; $d=8$

can go down once with α^2 from $\omega^1 = \mu^1+\mu^2$

can go down once with α^1 from $\omega^1 = \mu^1+\mu^2$

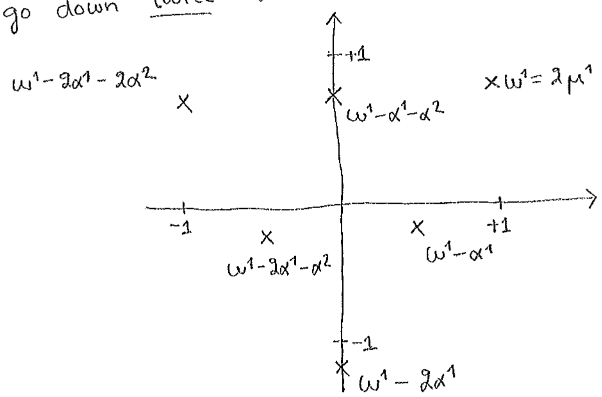
"adjoint" representation 8



$*\ (2,0)$; $d=6$

cannot operate with α^2 on $\omega^1 = 2\mu^1$

can go down twice with α^1 from $\omega^1 = 2\mu^1$.



Appendix: proof of the "main formula" from p. 17

(a) Denote an eigenvector with weight ω now with $|\omega\rangle$.
We know from p. 16 that $\mathcal{D}(E_k)|\omega\rangle \propto |\omega+\alpha\rangle$, $\mathcal{D}(E_{-k})|\omega\rangle \propto |\omega-\alpha\rangle$.
Let $N_{\pm\alpha,\omega}$ be the normalization constants: $\mathcal{D}(E_{\pm k})|\omega\rangle \equiv N_{\pm\alpha,\omega}|\omega\pm\alpha\rangle$.

(b) We denote the scalar product in V by $\langle\omega|\omega\rangle$. It follows that

$$N_{-\alpha,\omega} = \langle\omega-\alpha|\mathcal{D}(E_{-k})|\omega\rangle = \langle\omega-\alpha|\mathcal{D}(E_k)^\dagger|\omega\rangle = \langle\omega|\mathcal{D}(E_k)|\omega-\alpha\rangle^* = N^*_{\alpha,\omega-\alpha}.$$

(c) We need an important property of $[E_k, E_{-k}]$:

$$[H_i,[E_k,E_{-k}]] \overset{\text{Jacobi identity}}{=} -[E_k,[E_{-k},H_i]] - [E_{-k},[H_i,E_k]]$$
$$= [E_k,[H_i,E_{-k}]] - [E_{-k},[H_i,E_k]]$$
$$\overset{\text{p.15}}{=} -\alpha^k_i\,[E_k,E_{-k}] - \alpha^k_i\,[E_{-k},E_k] \overset{\text{antisymmetry}}{=} 0.$$

Therefore $[E_k, E_{-k}]$ must belong to the Cartan subalgebra:

$$[E_k, E_{-k}] = \sum_{j=1}^{m} \beta_j H_j .$$

(d) We can determine the coefficients β_j:

$$\beta_j = 2\,\mathrm{tr}\{H_j[E_k,E_{-k}]\} = 2\,\mathrm{tr}\{E_{-k}[H_j,E_k]\} = 2\alpha^k_j\,\mathrm{Tr}\{E_k^\dagger E_k\} = \alpha^k_j .$$

$$\Rightarrow\quad [E_k, E_{-k}] = \sum_{j=1}^{m} \alpha^k_j H_j .$$

(we had suppressed the index in points (a) and (b).)

This holds also for $\mathcal{D}(E_k)$, $\mathcal{D}(H_j)$.

(e) Now we find

$$\alpha^k\cdot\omega = \langle\omega|\sum_{j=1}^{m}\alpha^k_j\mathcal{D}(H_j)|\omega\rangle = \langle\omega|[\mathcal{D}(E_k),\mathcal{D}(E_{-k})]|\omega\rangle$$
$$= \langle\omega|\mathcal{D}(E_k)\mathcal{D}(E_{-k})|\omega\rangle - \langle\omega|\mathcal{D}(E_{-k})\mathcal{D}(E_k)|\omega\rangle = |\mathcal{D}(E_k)^\dagger|\omega\rangle|^2 - |\mathcal{D}(E_k)|\omega\rangle|^2$$
$$= |N_{-\alpha,\omega}|^2 - |N_{\alpha,\omega}|^2 = |N_{\alpha,\omega-\alpha}|^2 - |N_{\alpha,\omega}|^2 . \qquad (*)$$

(f) The definitions of q and p from p. 17 can be expressed as

$$\mathcal{D}(E_k)|\omega+p\alpha\rangle = 0 \quad ; \quad \mathcal{D}(E_{-k})|\omega-q\alpha\rangle = 0 .$$

These imply

$$|N_{\alpha,\omega+p\alpha}| = |N_{-\alpha,\omega-q\alpha}| \overset{(b)}{=} |N_{\alpha,\omega-(q+1)\alpha}| = 0 .$$

(g) Use now $(*)$ with:

$$\begin{aligned}
\omega\to\omega+p\alpha:&\quad \alpha\cdot(\omega+p\alpha) = |N_{\alpha,\omega+(p-1)\alpha}|^2 - 0\\
\omega\to\omega+(p-1)\alpha:&\quad \alpha\cdot(\omega+(p-1)\alpha) = |N_{\alpha,\omega+(p-2)\alpha}|^2 - |N_{\alpha,\omega+(p-1)\alpha}|^2\\
&\quad\vdots \qquad\qquad\qquad\qquad\qquad\qquad \vdots\\
\omega\to\omega-(q-1)\alpha:&\quad \alpha\cdot(\omega-(q-1)\alpha) = |N_{\alpha,\omega-q\alpha}|^2 - |N_{\alpha,\omega-(q-1)\alpha}|^2\\
\omega\to\omega-q\alpha:&\quad \alpha\cdot(\omega-q\alpha) = 0 - |N_{\alpha,\omega-q\alpha}|^2
\end{aligned}$$

sum together

$$(p+q+1)\,\alpha\cdot\omega + \alpha\cdot\alpha\underbrace{\left[\frac{p(p+1)}{2} - \frac{q(q+1)}{2}\right]}_{\frac{(p+q+1)(p-q)}{2}} = 0$$

$p+q+1 \neq 0 \Rightarrow \square$.