1.4 Weights and roots

* sometimes this very long concept is abbreviated as "irrep".

Outline: We would like to

(i) classify all irreducible representations* of a given group;

(ii) find a way to reduce a general representation into a "block diagonal" sum of irreducible reprensentations.

We have already defined one characteristic of a representation, its dimension d (cf. p. 10), but in general this is not sufficient for enumerating all cases.

Example: From quantum mechanics, we are already familiar with representations of $SU(2)$ [spin] and $SO(3)$ [rotations]: these are classified through a "quantum number" $s = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$ or $\ell = 0, 1, 2, \ldots$.
The corresponding dimension is $d = 2s+1$ or $d = 2\ell + 1$, i.e. the number of possible z-components.

Definitions: Let T^a be the generators of the fundamental representation (p. 9), F^a those of the adjoint one (p. 10), and $\mathcal{D}(T^a)$ those of a general representation.

Let us diagonalize simultaneously as many T^a as possible. We denote the diagonal ones by H_i, $i = 1, \ldots, m$, where m is called the rank of the group. If the T^a are hermitean (p. 11), the H_i are real. Since the H_i are diagonal, they satisfy

$$[H_i, H_j] = 0 \quad \Rightarrow \quad [\mathcal{D}(H_i), \mathcal{D}(H_j)] = 0 .$$

The H_i span the "maximal Abelian subalgebra" or the "Cartan subalgebra".

Examples:

* $SU(2) \Rightarrow H_1 = \frac{\sigma_3}{2} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \Rightarrow \text{rank} = 1.$

* $SU(3)$: $(T^a)^\dagger = T^a$, $\operatorname{tr} T^a = 0$, $\operatorname{tr}(T^a T^b) = \frac{\delta^{ab}}{2}$.

A possible choice is

$$H_1 = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad H_2 = \frac{1}{2\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

$\Rightarrow$ rank $= 2$.

There are in total $n^2 - 1 \overset{n=3}{=} 8$ generators, so 6 independent non-diagonal ones left over.

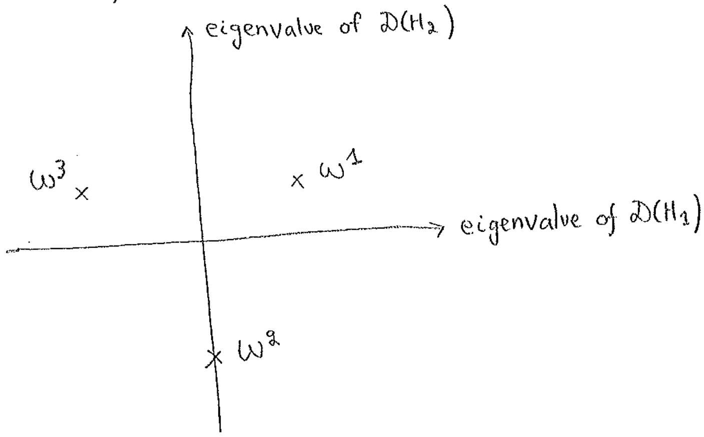
Weights: In a general representation, $D(H_i)$, $i=1,\dots,m$, are $d\times d$ matrices. Because they commute, they are diagonal; the diagonal components are the eigenvalues of the given $D(H_i)$. In total, we have then $m\times d$ eigenvalues.

We now define d weight vectors*, w^j, $j=1,\dots,d$, each of which has m components w^j_i, namely the eigenvalues ("weights") from a chosen row* j of $D(H_i)$, $i=1,\dots,m$.
The weight vectors can be drawn in m-dimensional space.

* a convenient ordering is defined below

Example: For H_1, H_2, of $SU(3)$ from p. 13 we get $(d=3)$

$$w^1 = \left(\tfrac{1}{2}, \tfrac{1}{2\sqrt{3}}\right), \quad w^3 = \left(-\tfrac{1}{2}, \tfrac{1}{2\sqrt{3}}\right), \quad w^2 = \left(0, -\tfrac{1}{\sqrt{3}}\right).$$



Ordering: We choose w^1 to be the weight vector with the largest component along the x-axis; if the same, then the largest along the y-axis; etc.
If so ordered, we say that $w^1 > w^2 > w^3$.

Physical interpretation: The simultaneously diagonalized operators $D(H_i)$ represent physical observables (e.g. S_z for $SU(2)$).
The corresponding eigenvalues are possible values of measurements. The weight vectors w^j are the possible eigenstates with respect to these physical observables. In a d-dimensional representation, there are d eigenstates, some of which might be degenerate (i.e. with the same eigenvalues).

Roots:

Let E_k be the remaining (non-diagonal) generators of the fundamental representation. We choose them as linear combinations of the T^a, with in general complex coefficients, so that they satisfy

$$[H_i, E_k] = \alpha_i^k E_k, \quad i=1,\dots,m \ ; \quad k=1,\dots,\dim - m.$$

The root vectors α^k live in the same space as the w_i, and have the α_i^k as their components, i.e. $\alpha^k = (\alpha_1^k, \dots, \alpha_m^k)$.

Properties:

The way that E_k have been defined, they are not necessarily Hermitean! The α_i^k are real (see below) $\Rightarrow$

$$\alpha_i^k E_k^\dagger = (H_i E_k - E_k H_i)^\dagger = E_k^\dagger H_i - H_i E_k^\dagger$$

$$\Leftrightarrow [H_i, E_k^\dagger] = -\alpha_i^k E_k^\dagger .$$

It follows that if α^k is a root vector, $-\alpha^k$ is also so.

We choose the normalization as $\mathrm{tr}(E_k E_\ell^\dagger) = \frac{\delta_{k\ell}}{2}$.

Example:

$SU(3)$

$$\{E_k\} = \left\{ \frac{1}{\sqrt2}\begin{pmatrix}0&1&0\\0&0&0\\0&0&0\end{pmatrix}, \frac{1}{\sqrt2}\begin{pmatrix}0&0&1\\0&0&0\\0&0&0\end{pmatrix}, \frac{1}{\sqrt2}\begin{pmatrix}0&0&0\\0&0&1\\0&0&0\end{pmatrix}, \frac{1}{\sqrt2}\begin{pmatrix}0&0&0\\1&0&0\\0&0&0\end{pmatrix}, \frac{1}{\sqrt2}\begin{pmatrix}0&0&0\\0&0&0\\1&0&0\end{pmatrix}, \frac{1}{\sqrt2}\begin{pmatrix}0&0&0\\0&0&0\\0&1&0\end{pmatrix} \right\}.$$

In other words, $(E_k)_{ij} = \frac{1}{\sqrt2}\delta_{im}\delta_{jn}$, with $m \neq n$.

With this choice, we can work out the commutator:

$$\begin{aligned}
[H_i, E_k]_{ab} &= (H_i)_{aj}(E_k)_{jb} - (E_k)_{aj}(H_i)_{jb} \\
&= \frac{1}{\sqrt2}\Big[\underbrace{(H_i)_{aj}}_{(H_i)_{aa}\delta_{aj}} \delta_{jm}\delta_{bn} - \delta_{am}\delta_{jn} \underbrace{(H_i)_{jb}}_{(H_i)_{bb}\delta_{jb}} \Big] \\
&= \frac{1}{\sqrt2}\big[(H_i)_{aa} - (H_i)_{bb} \big]\delta_{am}\delta_{bn} \\
&= \underbrace{\big[(H_i)_{aa} - (H_i)_{bb} \big]}_{\alpha_i^k \ !} (E_k)_{ab}
\end{aligned}$$

Why $\alpha^k \in \mathbb{R}$?

Let us consider the definition of weights (as eigenvalues of $H_i = T^i$) in the adjoint representation (p.10):

$$\begin{aligned}
\mathcal{D}(H_i)\, v &= F^i v \overset{\text{p.10}}{=} T^b (F^i)^{bc} v^c \\
&\overset{\text{p.10}}{=} T^b(-i f^{ibc}) v^c = i f^{icb} v^c T^b \\
&\overset{\text{p.6}}{=} [T^i, T^c] v^c = [H_i, v] .
\end{aligned}$$

In other words, if $[H_i, E_k] = \alpha_i^k E_k$, then

$$\mathcal{D}_{adj}(H_i) E_k = \alpha_i^k E_k$$

$\Rightarrow$ roots $\equiv$ weights of the adjoint representation!

$\Rightarrow \ \alpha^k \in \mathbb{R}^m$.

Relationship: Let v^j be one of the eigenvectors of $\mathcal{D}(H_i)$, i.e.

$$\mathcal{D}(H_i)\, v^j = w_i^j\, v^j \quad (\text{no sum over } j)$$

Then

$$\mathcal{D}(H_i)\,\mathcal{D}(E_k)\, v^j = \mathcal{D}([H_i, E_k])\, v^j + \mathcal{D}(E_k)\mathcal{D}(H_i)\, v^j$$

$$= \alpha_i^k\, \mathcal{D}(E_k)\, v^j + w_i^j\, \mathcal{D}(E_k)\, v^j .$$

$\Rightarrow$ $\mathcal{D}(E_k)\, v^j$ is also an eigenvector (if it is $\neq 0$), with weight vector $w^j + \alpha^k$

$\Rightarrow$ $\mathcal{D}(E_k^\dagger)\, v^j$ is also an eigenvector (if it is $\neq 0$), with weight vector $w^j - \alpha^k$.

What we learned:

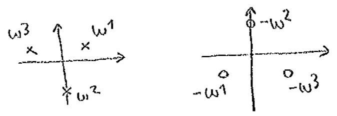
* all weights are of the form $w^l = w^j \pm \alpha^k$, all roots are of the form $\alpha^k = \pm(w^l - w^j)$.
* since roots are the weights of the adjoint representation, their number is the dimension of the adjoint representation, i.e. the dimension "dim" of the group manifold.
* among the "dim" roots, "m" are zero vectors, since $[H_l, H_j] = 0 \;\; \forall\, j = 1, \dots, m$.
* Positive roots*: $w^l - w^j$, with $l < j$ $(w^1 > w^2 > \dots)$.
* Simple roots: positive and not sums of other positive roots, i.e. linearly independent. There are "m" such roots, and we can find them from the set $\{w^l - w^{l+1}\}$.

* it is convenient to enumerate positive α^k with indices $k > 0$

Example: $SU(3)$, $w^1 = \left(\frac{1}{2}, \frac{1}{2\sqrt{3}}\right)$, $w^2 = \left(0, -\frac{1}{\sqrt{3}}\right)$, $w^3 = \left(-\frac{1}{2}, \frac{1}{2\sqrt{3}}\right)$.

As simple roots we can choose e.g. $\alpha^1 = w^1 - w^2 = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

$\alpha^2 = w^2 - w^3 = \left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$

All weights: 

Roots $\Rightarrow$