1.3 Representations

Starting point: We have defined invariance groups (p.3) through the properties of a transformation $v \mapsto Av$, with $v \in \mathbb{C}^n$.
The realization of the group structure through transformations in a vector space is called a representation, in this case the defining or fundamental representation. However, (infinitely) many other representations can also be defined.

Vector space: The basic characteristics of a vector space are:

* $v, w \in V \Rightarrow \alpha v + \beta w \in V \quad \forall \alpha, \beta \in \mathbb{C}$.
* the vectors $v_1, \dots, v_k$ are linearly independent, if
$$\sum_{i=1}^{k} \alpha_i v_i = 0 \Rightarrow \alpha_1 = \dots = \alpha_k = 0 .$$
* the dimension of a vector space is d, if there are at most d linearly independent vectors.
* every vector $v \in V$ can be expressed as
$$v = \sum_{i=1}^{d} v_i \hat{e}_i ,$$
where $\hat{e}_i$ are the basis vectors, and $v_i \in \mathbb{C}$ the components of the vector in this basis.

Linear mappings: We consider transformations with the following properties:

* a mapping $L: V \to V$ is linear, if it respects the structure of the vector space:
$$L(\alpha v + \beta w) = \alpha L(v) + \beta L(w) .$$
* if L^{-1} exists, the linear mappings form a group, with multiplication
$$(L_1 \cdot L_2)(v) \equiv L_1(L_2(v)) .$$
This group is denoted by $GL(V)$ or $GL(d, \mathbb{C})$ (p.3).
* often we call these transformations "operators" acting on V.
* if we express v in a given basis, then
$$L(v) = L\Big(\sum_i v_i \hat{e}_i\Big) = \sum_i v_i \underbrace{L(\hat{e}_i)}_{\boxed{\equiv \sum_j \hat{e}_j L_{ji}}}$$
$$= \sum_j \Big(\sum_i L_{ji} v_i\Big) \hat{e}_j .$$
The complex numbers L_{ji} form a matrix in this basis.

Definition: A (linear) representation of the group G is a homomorphism $D: G \to GL(V)$, $g \mapsto D(g) \in GL(V)$, with $D(g_1 \cdot g_2) = D(g_1) \cdot D(g_2)$.
The dimension of the representation is the dimension of V, d.

Remark: The dimension d is independent of n – only for the fundamental representation (p.9) do we have $d = n$.
Both of these are independent of the number of coordinates parametrizing the group elements, dim.

Examples: We define two important representations, which are different from the fundamental representation.

* Conjugated (or complex conjugated) representation:

$$d = n$$
$$D(g) \equiv g^*$$
$$D(g_1 g_2) = (g_1 \cdot g_2)^* = g_1^* \cdot g_2^* = D(g_1) \cdot D(g_2) \qquad \text{OK!}$$

notation: n^* or $\bar{n}$

generators: $[\exp(i\theta^a T^a)]^* = \exp(-i\theta^a [T^a]^*) \Rightarrow D(T^a) = -(T^a)^*$.

structure constants: $[T^a, T^b] = i f^{abc} T^c \Rightarrow [(-T^a)^*, (-T^b)^*] = i \underbrace{f^{abc}}_{\text{the same!}} (-T^c)^*$.

More generally, the structure constants are independent of the representation.

* adjoint representation:

The basis vectors of V are now matrices, namely T^a!

$$d = \dim$$
$$v = \sum_{a=1}^{\dim} v_a T^a \; ; \quad v' \equiv D(g)(v) \equiv g\, v\, g^{-1} .$$

This transformation is linear $[D(g)(v+w) = D(g)(v) + D(g)(w)]$ and bijective $[v = g^{-1} v' g.]$.

The generators can be found by considering "small" transformations, $g = e^{i\theta^a T^a}$, $\theta^a \ll 1$.
The transformed vector can be written as

$$v' = \sum_b (v')^b T^b , \qquad (v')^b = v^b + i\theta^a (F^a)^{bc} v^c + O(\theta^2).$$

Here the generators F^a are dim×dim-matrices, whose components are $(F^a)^{bc} = -i f^{abc}$.

Once again, it can be verified (cf. exercise 3.1) (starting from the Jacobi identity) that

$$[F^a, F^b] = i f^{abc} F^c .$$

<u>Further definitions:</u> * two representations D_1, D_2 are <u>equivalent</u>, if there is a linear invertible mapping S, called a <u>similarity transformation</u>, so that

$$D_2(g) = S\, D_1(g)\, S^{-1} \quad \forall\, g\,.$$

This corresponds effectively to a basis transformation on V:

$$D_2(g) S = S D_1(g)$$

$$\begin{tikzcd} V_1 \arrow[r, shift left, "S"] \arrow[d, "D_1"'] & V_2 \arrow[l, shift left, "S^{-1}"] \arrow[d, "D_2"] \\ V_1 \arrow[r, shift left, "S"] & V_2 \arrow[l, shift left, "S^{-1}"] \end{tikzcd}$$

For basis vectors: $\hat{e}_i^{(2)} = S\, \hat{e}_i^{(1)}$

For generators: $T_2^a = S\, T_1^a\, S^{-1} \quad \forall\, a = 1, \dots, \dim$.

* the <u>kernel</u> of a representation is defined as

$$\ker(D) = \{ g \in G \mid D(g) = \mathbb{1}_{d \times d} \}\,.$$

If $\ker(D) = \{e\}$, then the representation is "<u>faithful</u>".

For a faithful representation, $g_1 \neq g_2 \Rightarrow D(g_1) \neq D(g_2)$.

[Assume $D(g_1) = D(g_2) \Rightarrow D(g_1 g_2^{-1}) = D(g_1) D(g_2^{-1}) = D(g_2) D(g_2^{-1})$
$= D(g_2 g_2^{-1}) = D(e) = \mathbb{1}$
$\overset{\text{faithful}}{\Rightarrow} g_1 g_2^{-1} = e \Rightarrow g_1 = g_2$.]

* if the matrices $D(g)$ are unitary, the representation is <u>unitary</u>. (The group G itself does not need to be unitary.)

It turns out, that for all finite or "compact" (see margin) continuous groups, a positive definite scalar product can be defined, which renders D unitary.

[Let $\langle v | w \rangle \equiv \sum_{i=1}^{d} v_i^* w_i$. We define*

$$\langle\langle v | w \rangle\rangle \equiv \frac{1}{|G|} \sum_{g \in G} \langle D(g) v \mid D(g) w \rangle\,.$$

Then $\langle\langle v \mid D^\dagger(g') D(g') w \rangle\rangle \equiv \langle\langle D(g') v \mid D(g') w \rangle\rangle$

$$= \frac{1}{|G|} \sum_{g \in G} \langle D(g) D(g') v \mid D(g) D(g') w \rangle = \frac{1}{|G|} \sum_{g \in G} \langle D(g \cdot g') v \mid D(g \cdot g') w \rangle$$

$$= \frac{1}{|G|} \sum_{g'' \in G} \langle D(g'') v \mid D(g'') w \rangle = \langle\langle v | w \rangle\rangle \Rightarrow D^\dagger(g') D(g') = \mathbb{1}. \ \square$$]

In the following, we mostly study unitary representations. The corresponding generators are <u>Hermitean</u>.

* The notation here assumes a finite group, with $|G|$ denoting the number of group elements. For continuous groups, the sum is replaced by integration, with $dg = d\, g \cdot g' = d\, g' \cdot g$ being a left and right-invariant "<u>Haar measure</u>". A group is compact, if we can normalize $\int_G dg = 1$.

Further examples: * direct sum of representations

Let D_1, D_2 be representations of the group G in V_1, V_2. By constructing the direct sum $V_1 \oplus V_2 = \{(v_1, v_2) \mid v_i \in V_i\}$, with $(v_1, v_2) \equiv (v_1, 0) + (0, v_2)$, we find a new representation:

$$D_1 \oplus D_2 (g) (v_1, v_2) \equiv (D_1(g) v_1, 0) + (0, D_2(g) v_2).$$

As basis vectors we can choose $\{(\hat{e}_i, 0)\}$ together with $\{(0, \hat{e}_j)\}$:

$$(v, v') = \left(\sum_{i=1}^{d_1} v_i \hat{e}_i, \sum_{j=1}^{d_2} v'_j \hat{e}_j \right) = \sum_{i=1}^{d_1} v_i (\hat{e}_i, 0) + \sum_{j=1}^{d_2} v'_j (0, \hat{e}_j).$$

The dimension of the representation is thus $d_{D_1 \oplus D_2} = d_1 + d_2$. In the chosen basis, the matrix corresponding to $D_1 \oplus D_2$ is block diagonal: $\left(\begin{array}{c|c} D_1 & 0 \\ \hline 0 & D_2 \end{array} \right)$.

A representation is irreducible, if there are no invariant subspaces, i.e. the matrices cannot be written in a block diagonal form.

A representation is fully reducible, if D is equivalent to $D_1 \oplus D_2 \oplus \dots$, where the D_i are irreducible.

* direct product of representations

By constructing the direct product $V_1 \otimes V_2 \equiv \{(v_1, v_2) \mid v_i \in V_i\}$, with

$$(\alpha_1 v_1 + \beta_1 w_1, v_2) = \alpha_1 (v_1, v_2) + \beta_1 (w_1, v_2)$$
$$(v_1, \alpha_2 v_2 + \beta_2 w_2) = \alpha_2 (v_1, v_2) + \beta_2 (v_1, w_2),$$

we find another new representation:

$$D_1 \otimes D_2 (g) (v_1, v_2) \equiv (D_1(g) v_1, D_2(g) v_2).$$

As basis vectors we can choose $\{(\hat{e}_i, \hat{e}_j)\}$:

$$(v, v') = \sum_{i=1}^{d_1} \sum_{j=1}^{d_2} v_i v'_j (\hat{e}_i, \hat{e}_j).$$

The dimension of the representation is $d_{D_1 \otimes D_2} = d_1 \cdot d_2$.

Outlook: The goal of reduction of representations, the topic of the next sections, is to write $D_1 \otimes D_2$ as a sum,

$$D_1 \otimes D_2 = D_a \oplus D_b \oplus D_c \oplus \dots$$

In other words, we want to identify all invariant subspaces.