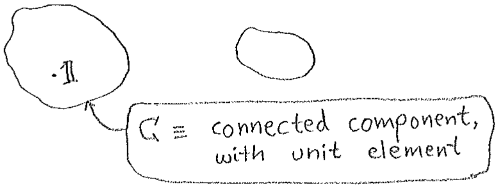
1.2 Lie group $\Leftrightarrow$ Lie algebra

Choice of coordinates: We consider an invariance group, with $A^\dagger \eta A = \eta$ ⟦and perhaps $\det A = 1$⟧.

Group manifold:



Claim: every $A \in G$ can be expressed as $A = \exp\left(i \sum_{a=1}^{\dim} \theta^a T^a\right)$, where $\theta^a \in \mathbb{R}$ and the "generators" T^a satisfy $(T^a)^\dagger \eta - \eta T^a = 0$ ⟦and $\mathrm{tr}\, T^a = 0$ if $\det A = 1$⟧. This representation is not unique.

Why is it so?

$\boxed{\text{"}\Leftarrow\text{"}}$ Let us denote $\Theta \equiv \sum_a \theta^a T^a$, and define a mapping $\mathbb{R}^{\dim} \to M(n, \mathbb{C})$ as $\{\theta^a\} \mapsto \exp(i\Theta)$. Then $\exp(i\Theta) \in G$.
To see this, let $f(\lambda) \equiv \exp(i\lambda\Theta)$ and $g(\lambda) \equiv [f(\lambda)]^\dagger \eta\, [f(\lambda)]$.

$$\Rightarrow \frac{d}{d\lambda} g(\lambda) = \frac{d}{d\lambda}\left[\exp(-i\lambda\Theta^\dagger)\,\eta\,\exp(i\lambda\Theta)\right]$$
$$= -i \exp(-i\lambda\Theta^\dagger)\,\underbrace{\left[\Theta^\dagger \eta - \eta\Theta\right]}_{0}\,\exp(i\lambda\Theta) = 0$$

$$\Rightarrow g(\lambda) = g(0) = \eta \quad \forall \lambda \quad \Rightarrow \quad f(\lambda) \in G \quad \forall \lambda .$$

⟦Furthermore, $\det[\exp(i\Theta)] = \prod_{a=1}^{n} \exp(i\lambda_a) = \exp(i\sum_a \lambda_a) = \exp(i\,\mathrm{tr}\,\Theta) = 1$.⟧

after diagonalization (though more general)

$\mathrm{tr}\,T^a = 0$

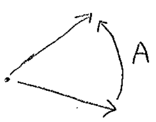
$\boxed{\text{"}\Rightarrow\text{"}}$ It is more difficult to show that every $A \in G$ can be reached, i.e. that the mapping is surjective, but we can give an argument. It follows from connectedness that

$$\forall A \in G \quad \exists f: [0,1] \to C,\ \lambda \mapsto f(\lambda), \quad \text{with } f(0) = \mathbb{1},\ f(1) = A.$$

For $\lambda \ll 1$, $f(\lambda) \equiv \mathbb{1} + \lambda i \Theta + \mathcal{O}(\lambda^2)$. Since $f(\lambda) \in G$,

$$[f(\lambda)]^\dagger \eta\, [f(\lambda)] = \eta \quad \Rightarrow \quad \Theta^\dagger \eta - \eta\Theta = 0 .$$

Now the matrix Θ can be represented as a linear combination of the matrices T^a $\Rightarrow \Theta = \sum_a \theta^a T^a$.



On the other hand, $f(1) = A$ can be viewed as a transformation: $v \mapsto Av$. Let us imagine that this "large" transformation is decomposed into a succession of "small" transformations:

$$A = f(1) = f(\tfrac{1}{2}) \cdot f(\tfrac{1}{2}) = \lim_{N\to\infty} \left[f\left(\tfrac{1}{N}\right)\right]^N$$
$$= \lim_{N\to\infty} \left[\mathbb{1} + \frac{i\Theta}{N} + \mathcal{O}\left(\frac{1}{N^2}\right)\right]^N$$
$$\overset{!}{=} \exp(i\Theta) . \qquad \text{"}\square\text{"} .$$

Lie-algebra: It is much simpler to study the matrices $\Theta = \sum_a \theta^a T^a$ than the group elements $\exp(i\Theta)$, because they form a real linear vector space, with T^a as basis vectors.
We call this a Lie algebra: group $SU(n) \rightarrow$ algebra $su(n)$, etc.
The notion of an algebra means that two operations can be defined, addition $\Theta + \Theta'$, as well as a multiplication.
The latter originates as follows.

Mapping of group operations onto algebra operations:

Consider the group elements $A = e^{i\theta^a T^a}$ and $B = e^{i\varphi^a T^a}$, where we have introduced the Einstein convention $\theta^a T^a \equiv \sum_{a=1}^{\dim} \theta^a T^a$.
The product belongs to G $\Rightarrow$ $A \cdot B \overset{!}{=} e^{i\xi^a T^a}$.
We expand on both sides to second order*:

*Higher orders are discussed on p. 8.

$$\underbrace{\left[\mathbb{1} + i\theta^a T^a - \tfrac{1}{2}\theta^a\theta^b T^a T^b\right]}_{A}\underbrace{\left[\mathbb{1} + i\varphi^a T^a - \tfrac{1}{2}\varphi^a\varphi^b T^a T^b\right]}_{B} = \mathbb{1} + i\xi^a T^a - \tfrac{1}{2}\xi^a\xi^b T^a T^b + \mathcal{O}(\theta,\varphi,\xi)^3$$

$$\Rightarrow \quad \mathbb{1} + i(\theta^a + \varphi^a)T^a - \tfrac{1}{2}(\theta^a\theta^b + \varphi^a\varphi^b + 2\theta^a\varphi^b)T^a T^b = \mathbb{1} + i\xi^a T^a - \tfrac{1}{2}\xi^a\xi^b T^a T^b + \mathcal{O}(\theta,\varphi,\xi)^3 .$$

Here we can write $T^a T^b = \frac{1}{2}\{T^a, T^b\} + \frac{1}{2}[T^a, T^b]$, and symmetrize the coefficients of the anticommutators, e.g. $2\theta^a\varphi^b\{T^a,T^b\} = (\theta^a\varphi^b + \theta^b\varphi^a)\{T^a,T^b\}$. So it follows that

$$\mathbb{1} + i(\theta^a + \varphi^a)T^a - \tfrac{1}{4}(\theta^a + \varphi^a)(\theta^b + \varphi^b)\{T^a,T^b\} - \tfrac{1}{2}\theta^a\varphi^b[T^a,T^b]$$
$$= \mathbb{1} + i\,\xi^a T^a - \tfrac{1}{4}\xi^a\xi^b\{T^a,T^b\} + \mathcal{O}(\theta,\varphi,\xi)^3 .$$

These can be equivalent only if $-\frac{1}{2}\theta^a\varphi^b[T^a,T^b]$ gives a second order correction to $i\xi^a T^a$!

So it must be that $[T^a, T^b] = i f^{abc}\, T^c$, where f^{abc} are called the structure constants of the group and of the corresponding algebra.

Inserting the structure constants, we find
$-\frac{1}{2}\theta^a\varphi^b[T^a,T^b] = -\frac{i}{2}\,\theta^d\varphi^b f^{dba}\, T^a$, and therefore

$$\xi^a = \theta^a + \varphi^a - \tfrac{1}{2}\,\theta^d\varphi^b f^{dba} + \mathcal{O}(\theta,\varphi)^3 .$$

Incidentally, we now see that $f^{abc} \in \mathbb{R}$.

Summary: When we multiply group elements, then on the algebra side we take sums but also commutators. The structure constants define how the commutators map onto the linear vector space spanned by the generators.

Properties of generators and structure constants

* the commutators appearing in a Lie algebra satisfy

  (i) $[X,Y] = -[Y,X]$ (antisymmetry),

  (ii) $[[X,Y],Z]+[[Y,Z],X]+[[Z,X],Y]=0$ (Jacobi identity).

* as always in a linear vector space, linear combinations of the T^a can also serve as basis vectors. The basis is orthogonal, if $\mathrm{tr}[T^aT^b] \propto \delta^{ab}$. In most cases $[O(n), SO(n), U(n), SU(n), \dots]$, we can also normalize it via
$$\mathrm{tr}[T^aT^b] = \frac{\delta^{ab}}{2}.$$

* the basis must also be complete: every element of the algebra can be expressed as a linear combination of the T^a. This leads to a completeness relation; for instance, for the generators of $SU(n)$ [or $su(n)$] (cf. exercise 2.4)
$$\sum_{a=1}^{n^2-1} T^a_{ij}T^a_{kl} = \frac{1}{2}\left(\delta_{il}\delta_{jk} - \frac{1}{n}\delta_{ij}\delta_{kl}\right).$$

* it follows from $\mathrm{tr}[T^aT^b] = \frac{\delta^{ab}}{2}$ and $[T^a,T^b] = if^{abc}T^c$ that
$$f^{abc} = -2i\,\mathrm{tr}\{[T^a,T^b]T^c\}.$$
From here it follows that f^{abc} is antisymmetric not only in $a \leftrightarrow b$, but also in $a \leftrightarrow c$ and $b \leftrightarrow c$ (exercise 2.1).
Moreover, from the Jacobi identity, it can be shown that (—"—)
$$f^{abd}f^{cde} + f^{bcd}f^{ade} + f^{cad}f^{bde} = 0 \quad \forall a,b,c,e \in \{1,\dots,\dim\}.$$

* if there is a generator which commutes with all the other generators, it generates an "Abelian subalgebra". If not, the algebra is "semisimple".

* example: $su(2)$; $\dim = 2^2-1 = 3$.

  Pauli matrices (p.4): $\sigma_1 = \begin{pmatrix}0&1\\1&0\end{pmatrix}$, $\sigma_2 = \begin{pmatrix}0&-i\\i&0\end{pmatrix}$, $\sigma_3 = \begin{pmatrix}1&0\\0&-1\end{pmatrix}$.

  Generators: $T^a = \frac{\sigma_a}{2}$; $\mathrm{tr}[T^aT^b] = \frac{\delta^{ab}}{2}$; $(T^a)^\dagger = T^a$; $\mathrm{tr}(T^a) = 0$.

  Commutator: $[T^a,T^b] = i\varepsilon^{abc}T^c$, ε^{abc} = Levi-Civita symbol.

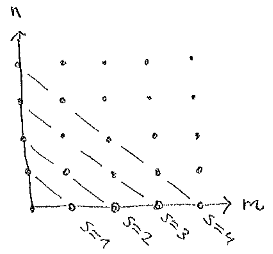
  There is no Abelian subalgebra.

Appendix: Campbell – Baker – Hausdorff formula

This relation shows generally how a multiplication of group elements can be reduced to operations within the algebra:

$$e^{tX} e^{tY} = e^{\sum_{n=1}^{\infty} t^n C_n(X,Y)} .$$

Let us determine C_1, C_2 (already on p. 6) and C_3.



$$e^{tX} = \sum_{n=0}^{\infty} \frac{t^n}{n!} X^n \quad ; \quad e^{tY} = \sum_{m=0}^{\infty} \frac{t^m}{m!} Y^m .$$

$$\Rightarrow e^{tX} e^{tY} = 1 + \underbrace{\sum_{s=1}^{\infty} \sum_{m=0}^{s} \frac{t^s}{m!(s-m)!} X^{s-m} Y^m}_{\equiv z} . \qquad (*)$$

$s = m+n$
$n = s-m$

We write $1 + z \equiv e^{\sum_{n=1}^{\infty} t^n C_n} \Rightarrow \sum_{n=1}^{\infty} t^n C_n = \ln(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \dots$

From $(*)$, identify z up to third order in t:

$$z = t(X+Y) + t^2\left(\frac{X^2}{2} + XY + \frac{Y^2}{2}\right) + t^3\left(\frac{X^3}{6} + \frac{X^2Y}{2} + \frac{XY^2}{2} + \frac{Y^3}{6}\right) + \dots$$

Then we have to work out z^2 and z^3, again to third order:

$$z^2 = t^2(X^2 + XY + YX + Y^2) + t^3\Big(\frac{X^3}{2} + X^2Y + \frac{XY^2}{2} + \frac{YX^2}{2} + YXY + \frac{Y^3}{2}$$
$$+ \frac{X^3}{2} + XYX + \frac{Y^2X}{2} + \frac{X^2Y}{2} + XY^2 + \frac{Y^3}{2}\Big) + \dots$$

$$z^3 = t^3\left(X^3 + X^2Y + XYX + XY^2 + YX^2 + YXY + Y^2X + Y^3\right) + \dots$$

Now we can combine the three terms:

$$z - \frac{z^2}{2} + \frac{z^3}{3} = t(X+Y)$$
$$+ t^2\Big(\cancel{\frac{X^2}{2}} + XY + \cancel{\frac{Y^2}{2}}$$
$$\cancel{-\frac{X^2}{2}} - \frac{XY}{2} - \frac{YX}{2} \cancel{-\frac{Y^2}{2}}\Big)$$
$$+ t^3\Big(\cancel{\frac{X^3}{6}} + \frac{X^2Y}{2} + \frac{XY^2}{2} + \cancel{\frac{Y^3}{6}}$$
$$\cancel{-\frac{X^3}{2}} - \frac{3X^2Y}{4} - \frac{3XY^2}{4} \cancel{-\frac{Y^3}{2}} - \frac{XYX}{2} - \frac{YXY}{2} - \frac{YX^2}{4} - \frac{Y^2X}{4}$$
$$\cancel{+\frac{X^3}{3}} + \frac{X^2Y}{3} + \frac{XY^2}{3} + \cancel{\frac{Y^3}{3}} + \frac{XYX}{3} + \frac{YXY}{3} + \frac{YX^2}{3} + \frac{Y^2X}{3}\Big) + \dots$$

Then identify the commutators:

$$C_1 = X + Y$$
$$C_2 = \frac{1}{2}[X,Y]$$
$$C_3 = \frac{1}{12}\left(X^2Y + XY^2 - 2XYX - 2YXY + YX^2 + Y^2X\right)$$
$$= \frac{1}{12}\left([X,[X,Y]] + [Y,[Y,X]]\right) .$$