Advanced Concepts of Theoretical Physics (M. Laine, ExWi-117)

Organization:

* general setup: lectures < script < literature

script → exercises; literature → exercises; lectures → exercises; lectures → exam; exercises → exam

* script: · via ILIAS
* exercises:
  · via ILIAS
  · ask questions in tutorials
  · model solutions later via ILIAS

1. Group theory

1.1 Basic concepts

Why is it relevant for physics?

Laws of nature turn out to display many invariances. The set of transformations leaving $\hat{H}$ or $\mathcal{L}$ invariant forms a group. We also observe symmetries in "solutions", e.g. eigenstates of $\hat{H}$. Such state vectors belong to a representation of a group. One invariance (e.g. Lorentz symmetry) has many representations (e.g. $s = 0, \frac{1}{2}, 1, \ldots$)

Basic definitions: A group G is a set $\{g_1, g_2, \ldots\}$ with "multiplication" "$\cdot$", so that

(i) $g_1, g_2 \in G \Rightarrow g_1 \cdot g_2 \in G$ [closed]

(ii) $g_1 \cdot (g_2 \cdot g_3) = (g_1 \cdot g_2) \cdot g_3$ [associative]

(iii) $e \cdot g = g \cdot e = g$ [$\exists$ unit element]

(iv) $g^{-1} \cdot g = g \cdot g^{-1} = e$ [$\exists$ inverse element]

A group is Abelian if $g_1 \cdot g_2 = g_2 \cdot g_1 \ \forall\, g_1, g_2 \in G$, otherwise non-Abelian.

A subgroup $H \subset G$ exists, if $h_1, h_2 \in H \Rightarrow h_1 \cdot h_2 \in H$ & $h \in H \Rightarrow h^{-1} \in H$.

Two groups G, G' are isomorphic ($G \cong G'$)*, if there is a bijection $f: G \to G'$ with $f(g_1) \cdot f(g_2) = f(g_1 \cdot g_2)$.

*If f is not a bijection, but $f(g_1) \cdot f(g_2) = f(g_1 \cdot g_2)$, it is a homomorphism.

Examples:

* $\mathbb{Z}_2 = \{e, a\}$, defined through a multiplication table:

	e	a
e	e	a
a	a	e

* $S_2 = \{$permutations of two elements$\}$:

$$e = \begin{pmatrix} 1 \to 1 \\ 2 \to 2 \end{pmatrix}, \quad a = \begin{pmatrix} 1 \to 2 \\ 2 \to 1 \end{pmatrix}.$$

* Clearly $\mathbb{Z}_2 \cong S_2$.

<u>Continuous groups</u>: If the group elements can be parametrized through n real coordinates, $g(x)$ with $x \in \mathbb{R}^n$, then we speak of a <u>group of dimension n</u>.

If we inspect the multiplication of two elements,

$$g(x) \cdot g(y) = g(z) \quad , \quad \text{with} \quad z = f_1(x,y) ,$$

and the inverse of an element,

$$[g(x)]^{-1} = g(w) \quad , \quad \text{with} \quad w = f_2(x) ,$$

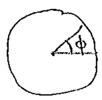
and the functions $f_{1,2}$ are continuous, then G is continuous.

If $f_{1,2}$ are <u>analytic</u>, we speak of a <u>Lie group</u>. We are mostly concerned with Lie groups here.

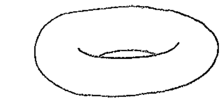
A space with a coordinate system is called a <u>manifold</u>. In general, a manifold requires several coordinate systems for its full mapping. Only locally does it look like a part of $\mathbb{R}^n$.

<u>Examples</u>:

	group	multiplication	unit element	dimension
*	$M(n,\mathbb{R})$ = $n \times n$ matrices, with elements $\in \mathbb{R}$	$+$	$0_{n \times n}$	n^2
*	$U(1) = \{z \in \mathbb{C} \mid \lvert z \rvert = 1\}$ parametrization: $z = e^{i\phi}$, $\phi \in [0, 2\pi)$	$\times$	1	1
*	T^n = n-torus $= U(1) \times \ldots \times U(1)$, $g = (e^{i\phi_1}, \ldots, e^{i\phi_n})$	$\times$	$(1, \ldots, 1)$	n



T^2



<u>Matrix groups</u>: Here multiplication $\equiv$ matrix multiplication. The existence of an inverse requires that $\det \neq 0$.

* $GL(n,\mathbb{R}) = \{ A \in M(n,\mathbb{R}) \mid \det A \neq 0 \}$; $\dim GL(n,\mathbb{R}) = n^2$
* $GL(n,\mathbb{C}) = \{ A \in M(n,\mathbb{C}) \mid \det A \neq 0 \}$; $\dim GL(n,\mathbb{C}) = 2n^2$

Note: even if $\dim GL(n,\mathbb{R}) = \dim M(n,\mathbb{R})$, the choice of coordinates is much more difficult, as the constraint $\det A \neq 0$ removes a complicated hypersurface from $\mathbb{R}^n$.

* $SL(n,\mathbb{R}) = \{ A \in GL(n,\mathbb{R}) \mid \det A = 1 \}$; $\dim SL(n,\mathbb{R}) = n^2 - 1$
* $SL(n,\mathbb{C}) = \{ A \in GL(n,\mathbb{C}) \mid \det A = 1 \}$; $\dim SL(n,\mathbb{C}) = 2(n^2 - 1)$

Note: <u>two</u> coordinates are eliminated by the constraints $\mathrm{Re} \det A = 1$, $\mathrm{Im} \det A = 0$.

Invariance groups: The matrix groups GL, SL have subgroups that are very important in physics. Let us inspect the scalar product

$$(v, w) \equiv \sum_{i,j=1}^{n} v_i^* \eta_{ij} w_j \quad ; \quad v, w \in \mathbb{C}^n .$$

(v_i, w_i are the components)

Now we require that the scalar product remains invariant in the transformation:

$$(Av, Aw) = \sum_{i,j,k,l=1}^{n} A_{ij}^* v_j^* \eta_{ik} A_{kl} w_l \equiv (v, w) \quad \forall v, w$$

$$\Rightarrow \quad \sum_{i,k=1}^{n} A_{ij}^* \eta_{ik} A_{kl} = \eta_{jl} \quad \forall j, l \in \{1, \dots, n\}$$

$$\Leftrightarrow \quad A^+ \eta A = \eta$$

$A^+ \equiv (A^*)^T \; ; \; [A^+]_{ji} \equiv A_{ij}^*$

"metric tensor"

Such transformations build a (sub)group:

(i) $A^+\eta A = \eta \quad \& \quad B^+\eta B = \eta$

$$\Rightarrow (AB)^+ \eta AB = B^+ \underbrace{A^+ \eta A}_{\eta} B = B^+ \eta B = \eta .$$

(iii) $e = \mathbb{1}_{n\times n} \; ; \quad \mathbb{1} \eta \mathbb{1} = \eta .$

(iv) $A^+ \eta A = \eta$

multiply from the right with A^{-1}

$\Rightarrow \quad A^+ \eta = \eta A^{-1}$

multiply from the left with $(A^{-1})^+$

$\Rightarrow \quad \eta = (A^{-1})^+ \eta (A^{-1}) \quad \Rightarrow \quad A^{-1}$ belongs to subgroup. □

Examples:

* $O(n) = \{ A \in GL(n, \mathbb{R}) \mid \eta = \mathbb{1}_{n\times n} \}$ "orthogonal matrices"

  $A^* = A \Rightarrow A^+ = A^T \Rightarrow A^T A = \mathbb{1}$

  $\dim O(n) = \frac{1}{2} n(n-1)^*$

* $SO(n) = \{ A \in O(n) \mid \det A = 1 \}$ "special orthogonal matrices"

  $\dim SO(n) = \frac{1}{2} n(n-1)$

* $U(n) = \{ A \in GL(n, \mathbb{C}) \mid \eta = \mathbb{1}_{n\times n} \}$ "unitary matrices"

  $\dim U(n) = n^2$

* $SU(n) = \{ A \in U(n) \mid \det A = 1 \}$ "special unitary matrices"

  $\dim SU(n) = n^2 - 1$

* $O(3,1) = \{ A \in GL(4, \mathbb{R}) \mid \eta = \mathrm{diag}(-1,1,1,1) \}$ "Lorentz group"

  $\dim O(3,1) = \dim O(4) = 6$

* $(A^T A)_{ij} = \delta_{ij}$

$i = j \Rightarrow n$ constraints

$i > j \Rightarrow \frac{n(n-1)}{2}$ constraints

$i < j \Rightarrow$ nothing new

$\Rightarrow \dim = n^2 - n - \frac{n(n-1)}{2}$.

All possible Lie groups were classified by Cartan. In addition to SO(n) and SU(n), they include "symplectic" groups, Sp(2n), and "exceptional" groups, G_2, F_4, E_6, E_7, E_8, where the index is "rank".

SU(2) in detail: Let us work out a full parametrization of this important example.

$$A^{\dagger} = \begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix} \quad ; \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$A^{\dagger}A = \begin{pmatrix} |a|^2+|c|^2 & a^*b+c^*d \\ b^*a+d^*c & |b|^2+|d|^2 \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \& \quad \det A = 1$$

$$\Rightarrow \begin{cases} |a|^2+|c|^2 = 1 \\ |b|^2+|d|^2 = 1 \end{cases} \quad \& \quad \begin{cases} b^*a+d^*c = 0 \\ ad-bc = 1 \end{cases}$$

(i) $\boxed{a \neq 0}$ $\Rightarrow b^* = -\frac{d^*c}{a} \Rightarrow b = -\frac{dc^*}{a^*}$

$$\Rightarrow ad-bc = \frac{d}{a^*}\underbrace{(|a|^2+|c|^2)}_{1} = 1 \quad \Rightarrow d = a^* \quad \Rightarrow b = -c^*$$

$$\Rightarrow A = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} \quad \text{with } |a|^2+|b|^2 = 1.$$

(ii) $\boxed{a = 0}$ $\Rightarrow |c| = 1 \Rightarrow d^* = 0 \Rightarrow |b| = 1 \Rightarrow c = -\frac{1}{b} = -\frac{bb^*}{b} = -b^*$

$$\Rightarrow A = \begin{pmatrix} 0 & b \\ -b^* & 0 \end{pmatrix} \quad \text{with } |b|^2 = 1.$$

Let us define $x_1 \equiv \mathrm{Re}\, a$, $x_2 \equiv \mathrm{Im}\, a$, $x_3 \equiv \mathrm{Re}\, b$, $x_4 \equiv \mathrm{Im}\, b$

$$\Rightarrow A = \begin{pmatrix} x_1+ix_2 & x_3+ix_4 \\ -x_3+ix_4 & x_1-ix_2 \end{pmatrix} \quad \text{with } x_1^2+x_2^2+x_3^2+x_4^2 = 1.$$

A manifold called an n-sphere, S^n, is defined as

$$S^n \equiv \left\{ (x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} \;\middle|\; \sum_{i=1}^{n+1} x_i^2 = 1 \right\}.$$

So we see that geometrically and topologically, SU(2) is like S^3 (connected, simply connected, ...).

Remarks:

* Often it is nice to employ the Pauli matrices
$$\sigma_0 \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \sigma_1 \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 \equiv \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then
$$A = x_1\sigma_0 + i\left[x_4\sigma_1 + x_3\sigma_2 + x_2\sigma_3\right] \underset{\substack{x_1 \to y_0 \\ x_4 \to y_1 \\ x_3 \to y_2 \\ x_2 \to y_3}}{=} y_0\sigma_0 + i\sum_{k=1}^{3} y_k\sigma_k.$$

* Geometrically and topologically, U(1) is like S^1. It turns out that S^1 and S^3 are the only spheres which can serve as group manifolds. In contrast, every torus T^n has a group structure (p. 2).

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