Exercise 1: In the lecture we have defined a "Euclidean" propagator, $D_E(\tau,\tau') \equiv A^{-1}(\tau,\tau')$, as the solution of the following differential equation:

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}+\omega^2\right)D_E(\tau,\tau') = \frac{\hbar}{m}\,\delta(\tau-\tau' \,\mathsf{mod}\, \beta\hbar)\;.$$

Here ω^2 is the usual parameter of the harmonic oscillator.

(a) Determine $D_E(\tau,0)$ for $0\le\tau\le\beta\hbar$.
[*Hint:* Solve the equation first at $0<\tau<\beta\hbar$. The solution contains two free parameters. They can be fixed for instance from the periodicity $D_E(0^+,0) = D_E(\beta\hbar^-,0)$, and from the boundary condition $D_E(0^+,0) \overset{\text{lecture}}{=} A^{-1}(\tau,\tau) \overset{\text{lecture}}{=} I/(m\beta) \overset{\text{lecture}}{=} \frac{\hbar}{2m\omega}\coth\frac{\beta\hbar\omega}{2}$.]
[*Answer:* $D_E(\tau,0) = \frac{\hbar}{2m\omega}\frac{\cosh(\frac{\beta\hbar}{2}-\tau)\omega}{\sinh\frac{\beta\hbar\omega}{2}}$.]

(b) Compare the result with a direct computation of

$$G_E(\tau,0) \;\equiv\; \frac{\mathsf{tr}[e^{-\beta\hat{H}}\hat{x}(\tau)\hat{x}(0)]}{\mathsf{tr}[e^{-\beta\hat{H}}]}\;,\quad 0<\tau<\beta\hbar\;,$$

where $\hat{x}(\tau) := e^{\frac{\tau\hat{H}}{\hbar}}\,\hat{x}\,e^{-\frac{\tau\hat{H}}{\hbar}}$ is a Heisenberg operator with $it\to\tau$, and $1/\,\mathsf{tr}[e^{-\beta\hat{H}}] = 2\sinh(\beta\hbar\omega/2)$. [*Hint:* $\langle n|\hat{x}|n'\rangle = \sqrt{\frac{\hbar}{2m\omega}}\Big(\sqrt{n'}\,\delta_{n,n'-1}+\sqrt{n}\,\delta_{n,n'+1}\Big)$.]

Exercise 2: Make use of the saddle point approximation, in order to derive the Stirling formula

$$n! = \left(\frac{n}{e}\right)^n\sqrt{2\pi n}\left[1+\frac{1}{12n}+\mathscr{O}\Big(\frac{1}{n^2}\Big)\right]\;.$$

[*Hint:* $n! = \int_0^\infty \mathrm{d}z\, z^n e^{-z} = \int_0^\infty \mathrm{d}z\, e^{-z+n\ln z}$.]

Exercise 3: Let $V(\phi)\ge 0$ be a non-negative smooth potential, and $\bar\phi(\tau)$ a solution of the Euclidean equation of motion

$$m\frac{\mathrm{d}^2\bar\phi}{\mathrm{d}\tau^2} = V'(\bar\phi)\;.$$

(a) Show that $\bar\phi$ fulfils $\frac{m}{2}(\frac{\mathrm{d}\bar\phi}{\mathrm{d}\tau})^2 - V(\bar\phi) = -E = \text{const}$, where E is positive in the case of periodic movement.

(b) Determine the period $P(E)$ as a function of E.
[*Answer:* $P = 2\int_{\bar\phi_1}^{\bar\phi_2}\frac{\mathrm{d}\bar\phi}{\sqrt{2[V(\bar\phi)-E]/m}}$, where $\bar\phi_i$ are the turning points, i.e. $V(\bar\phi_i)=E$.]

(c) Determine the classical action $\bar S_E^{(1+\bar 1)}$ of an instanton anti-instanton pair.
[*Answer:* $\bar S_E^{(1+\bar 1)} = 2\int_{\bar\phi_1}^{\bar\phi_2}\mathrm{d}\bar\phi\sqrt{2m[V(\bar\phi)-E]}+\beta\hbar E$.]

(d) Let now $V(\phi)\equiv\lambda(\phi^2-a^2)^2$. Show that $P(E)\to\infty$ for $E\to 0^+$.

(e) Verify that $\bar\phi(\tau)\equiv a\tanh(\omega(\tau_0-\tau)/2)$, with $m\omega^2\equiv 8\lambda a^2$, is a "half" solution in the limit $0\ll\tau_0\ll\beta\hbar$, i.e. satisfies the differential equation exactly, and the boundary conditions $\bar\phi(0)=+a$, $\bar\phi(\beta\hbar)=-a$ with exponentially small errors.