Exercise 1: Like in exercise 9.3, we consider the propagator of a harmonic oscillator:

$$\begin{aligned} K(\phi_2, t_2; \phi_1, t_1) &= \sqrt{\frac{m\omega}{2\pi i\hbar \sin[\omega(t_2 - t_1)]}} \\ &\times \exp\left\{ \frac{im\omega}{2\hbar \sin[\omega(t_2 - t_1)]} \Big[(\phi_1^2 + \phi_2^2) \cos[\omega(t_2 - t_1)] - 2\phi_1\phi_2 \Big] \right\} . \end{aligned}$$

Make use of this expression, in order to determine the wave functions $\psi_n(\phi) \equiv \langle \phi | n \rangle$ of the two lowest energy eigenstates, $n = 0$ and $n = 1$.
[*Answer:* $|\psi_0| = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} \exp\left(-\frac{m\omega\phi^2}{2\hbar}\right)$, $|\psi_1| = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} \left(\frac{2m\omega\phi^2}{\hbar}\right)^{\frac{1}{2}} \exp\left(-\frac{m\omega\phi^2}{2\hbar}\right)$.]

Exercise 2: Let E_0 be the ground state energy, and $Z \equiv \mathrm{tr}(e^{-\beta \hat{H}})$ the partition function.

(a) Show that the ground state energy can be obtained as $E_0 = -\lim_{\beta\to\infty} \dfrac{\ln Z}{\beta}$.

(b) Verify the identity $\langle 0 | \hat{A} | 0 \rangle = \lim_{\beta\to\infty} \dfrac{\mathrm{tr}(\hat{A}\, e^{-\beta \hat{H}})}{Z}$.

(c) Let $\hat{H}$ play the role of the operator $\hat{A}$. Show that this leads to $\langle 0 | \hat{H} | 0 \rangle = -\lim_{\beta\to\infty} \dfrac{\mathrm{d} \ln Z}{\mathrm{d}\beta}$. Does this conform with the result of point (a)?

Exercise 3: We inspect the "spectral representations" of two different Green's functions, "retarded" and "time-ordered":

$$D_R(t_2, t_1) \equiv \frac{\hbar}{m} \int_{-\infty}^{\infty} \frac{\mathrm{d}\nu}{2\pi} \frac{e^{-i\nu(t_2 - t_1)}}{\omega^2 - (\nu + i0^+)^2} , \quad D_T(t_2, t_1) := \frac{\hbar}{m} \int_{-\infty}^{\infty} \frac{\mathrm{d}\nu}{2\pi} \frac{e^{-i\nu(t_2 - t_1)}}{\omega^2 - \nu^2 - i0^+} .$$

(a) Show that both satisfy the differential equation $(\partial_t^2 + \omega^2) D = \frac{\hbar}{m} \delta(t - t')$.

(b) By inspecting the locations of the poles in the complex ν plane, argue that D_R vanishes for $t_2 < t_1$, whereas D_T is symmetric in $t_2 \leftrightarrow t_1$.

(c) Compute finally D_R and D_T explicitly, and demonstrate that their difference satisfies the homogeneous equation $(\partial_t^2 + \omega^2)(D_T - D_R) = 0$.
[*Answer:* $D_R = \frac{\hbar\, \theta(t_1 - t_1) \sin[\omega(t_2 - t_1)]}{m\omega}$, $D_T = \frac{i\hbar\, e^{-i\omega|t_2 - t_1|}}{2m\omega}$.]