ACTP sheet nr. 09

Exercise 1: In the lecture we found a result for the propagator of a free theory:

$$K(\phi_2,t_2;\phi_1,t_1) \;\equiv\; \langle\phi_2\,|e^{-\frac{i\hat{H}(t_2-t_1)}{\hbar}}|\,\phi_1\rangle = \sqrt{\frac{m}{2\pi i\hbar(t_2-t_1)}}\exp\biggl\{\frac{im(\phi_2-\phi_1)^2}{2\hbar(t_2-t_1)}\biggr\}\;.$$

Verify with this explicit expression that K has the following properties:

(a) "initial condition": $\lim\limits_{t_2\to t_1^+} K(\phi_2,t_2;\phi_1,t_1) = \delta(\phi_2-\phi_1)$.

(b) "equation of motion": K fulfils a Schrödinger equation with respect to ϕ_2 and t_2.

(c) "decomposition":

$$K(\phi_3,t_3;\phi_1,t_1) = \int_{-\infty}^{\infty}\mathrm{d}\phi_2\,K(\phi_3,t_3;\phi_2,t_2)K(\phi_2,t_2;\phi_1,t_1)\;,\quad t_1<t_2<t_3\;.$$

(d) "inverse":

$$\int_{-\infty}^{\infty}\mathrm{d}\phi_2\,\tilde{K}(\phi_3,t_1;\phi_2,t_2)K(\phi_2,t_2;\phi_1,t_1) = \delta(\phi_3-\phi_1)\;,$$

where $\tilde{K}(\phi_3,t_3;\phi_2,t_2)\equiv K^*(\phi_2,t_2;\phi_3,t_3)$ describes propagation "backwards in time".

Exercise 2: Let us consider a potential with a constant tilt, $V(\phi)=-F\phi$. At time t_1, the field has the value ϕ_1, at time t_2, ϕ_2.

(a) Determine the classical solution $\phi_{\text{cl}}(t)$ and the corresponding action $S_M[\phi_{\text{cl}}]$.

(b) Determine the quantum-mechanical propagator $K(\phi_2,t_2;\phi_1,t_1)$.

[*Answer:* $K=\sqrt{\frac{m}{2\pi i\hbar(t_2-t_1)}}\exp\Bigl\{\frac{i}{\hbar}\Bigl[\frac{m(\phi_2-\phi_1)^2}{2(t_2-t_1)}+\frac{F(\phi_1+\phi_2)(t_2-t_1)}{2}-\frac{F^2(t_2-t_1)^3}{24m}\Bigr]\Bigr\}$.]

Exercise 3: In the lecture we found a result for the propagator of a 1-dimensional harmonic oscillator:

$$\begin{aligned}K(\phi_2,t_2;\phi_1,t_1) \;&=\; \sqrt{\frac{m\omega}{2\pi i\hbar\sin[\omega(t_2-t_1)]}}\\ &\times\;\exp\biggl\{\frac{im\omega}{2\hbar\sin[\omega(t_2-t_1)]}\Bigl[(\phi_1^2+\phi_2^2)\cos[\omega(t_2-t_1)]-2\phi_1\phi_2\Bigr]\biggr\}\;.\end{aligned}$$

(a) Make use of $K(\phi_2,t_2;\phi_1,t_1)$ in order to determine $\mathrm{tr}\Bigl[e^{-\frac{i\hat{H}(t_2-t_1)}{\hbar}}\Bigr]$.
[*Hint:* Compute the trace in the $|\phi\rangle$ basis.] [*Answer:* $\frac{1}{2i\sin[\omega(t_2-t_1)/2]}$.]

(b) Extract from the result of point (a) the energy eigenvalues of the system.
[*Hint:* Compute the trace this time in the energy eigenbasis.]