ACTP	**sheet nr. 07**

Exercise 1: Consider the set $S = \{I, I_{\mathrm{P}}, I_{\mathrm{T}}, I_{\mathrm{PT}}\} \in$ O(3,1), where I is the identity transformation, I_{P} the space reflection, I_{T} the time reversal, and I_{PT} the spacetime reflection.

(a) Show that S is a subgroup of the Lorentz group.

(b) Construct the multiplication table for S.

(c) The group $\mathbb{Z}_2$ is the set $\mathbb{Z}_2 = \{e, a\}$ with the rule $a \cdot a = e$. Show that $S \cong \mathbb{Z}_2 \otimes \mathbb{Z}_2$.

Exercise 2: The general commutation relation of the Lorentz algebra reads

$$[J_{\mu\nu}, J_{\rho\sigma}] = -i\big(\eta_{\nu\rho}J_{\mu\sigma} - \eta_{\mu\rho}J_{\nu\sigma} - \eta_{\nu\sigma}J_{\mu\rho} + \eta_{\mu\sigma}J_{\nu\rho}\big)\ , \quad \eta = \text{diag}(-+++)\ .$$

Show that these imply

$$\begin{aligned} [J_{23}, J_{23}] &= 0\,, \qquad [J_{23}, J_{31}] = i\,J_{12}\ , \qquad [J_{23}, J_{12}] = -i\,J_{31}\ , \\ [J_{23}, J_{10}] &= 0\,, \qquad [J_{23}, J_{20}] = i\,J_{30}\ , \qquad [J_{23}, J_{30}] = -i\,J_{20}\ , \end{aligned} \tag{1}$$

and that the 2×2 matrices defined in the lecture satisfy these relations.

Exercise 3: Let γ_μ be the Dirac matrices, fulfilling the Clifford algebra $\{\gamma_\mu, \gamma_\nu\} = -2\eta_{\mu\nu}\mathbb{1}_{4\times4}$. Consider the standard represention

$$\gamma_0 = \begin{pmatrix} \mathbb{1}_{2\times2} & 0 \\ 0 & -\mathbb{1}_{2\times2} \end{pmatrix}\ , \qquad \gamma_k = \begin{pmatrix} 0 & -\sigma_k \\ \sigma_k & 0 \end{pmatrix}\ .$$

Verify that the 4×4 matrices $\mathscr{D}(J_{\mu\nu}) \equiv i[\gamma_\mu, \gamma_\nu]/4$ satisfy eq. (1).

Exercise 4: Let $\sigma_\mu \equiv (\mathbb{1}_{2\times2}, \sigma_k)$ and $\bar{\sigma}_\mu \equiv (\mathbb{1}_{2\times2}, -\sigma_k)$.

(a) Show that the generators of the fundamental representation of SL$(2,\mathbb{C})$ (from the lecture) can be written as

$$J_{\mu\nu} = \frac{i}{4}\left(\bar{\sigma}_\mu\sigma_\nu - \bar{\sigma}_\nu\sigma_\mu\right)\ . \tag{2}$$

(b) In the so-called Weyl representation we write the Dirac matrices as

$$\gamma_\mu = \begin{pmatrix} 0 & \bar{\sigma}_\mu \\ \sigma_\mu & 0 \end{pmatrix}\ .$$

If we consider the 4×4 representation matrices $\mathscr{D}(J_{\mu\nu}) = i[\gamma_\mu, \gamma_\nu]/4$ from exercise 3, how do the 2×2 matrices of eq. (2) make an appearance in them?