ACTP sheet nr. 06

Exercise 1: The symmetrizations and antisymmetrizations of the tensor method can be implemented with projectors. Let [for SU(3)]

$$[P_s]^{ij}_{kl} = \frac{1}{2}\left(\delta^i_k\delta^j_l + \delta^i_l\delta^j_k\right) \ , \quad [P_a]^{ij}_{kl} = \frac{1}{2}\left(\delta^i_k\delta^j_l - \delta^i_l\delta^j_k\right) \ .$$

Show that

(a)
$$[P_s]^{ij}_{kl} + [P_a]^{ij}_{kl} = \delta^i_k\,\delta^j_l \ , \qquad [P_s]^{ij}_{kl}\,[P_s]^{kl}_{mn} = [P_s]^{ij}_{mn},$$
$$[P_a]^{ij}_{kl}\,[P_a]^{kl}_{mn} = [P_a]^{ij}_{mn}, \qquad [P_s]^{ij}_{kl}\,[P_a]^{kl}_{mn} = 0 \,.$$

(b)
$$[P_s]^{ij}_{ij} = 6 \,, \qquad [P_a]^{ij}_{ij} = 3 \,.$$

How can we formulate the reduction $\mathbf{3}\otimes\mathbf{3} = \mathbf{6}\oplus\mathbf{3}^*$ with these projectors?

Exercise 2: Let then [for SU(3)]

$$[P_t]^{ik}_{jl} = \delta^i_l\delta^k_j - \frac{1}{3}\delta^i_j\delta^k_l \ , \quad [P_l]^{ik}_{jl} = \frac{1}{3}\delta^i_j\delta^k_l \ .$$

Show that

(a)
$$[P_t]^{ik}_{jl} + [P_l]^{ik}_{jl} = \delta^i_l\,\delta^k_j \ , \qquad [P_t]^{ik}_{jl}\,[P_t]^{lm}_{kn} = [P_t]^{im}_{jn} \ ,$$
$$[P_l]^{ik}_{jl}\,[P_l]^{lm}_{kn} = [P_l]^{im}_{jn} \ , \qquad [P_t]^{ik}_{jl}\,[P_l]^{lm}_{kn} = 0 \,.$$

(b)
$$[P_t]^{ij}_{ji} = 8 \,, \qquad [P_l]^{ij}_{ji} = 1 \,.$$

How can we formulate the reduction $\mathbf{3}^*\otimes\mathbf{3} = \mathbf{8}\oplus\mathbf{1}$ with these projectors?

Exercise 3: Let finally [for SU(3)]

$$\begin{aligned}
[P_A]^{ijk}_{mno} &\equiv \frac{1}{6}\left(\delta^i_m\,\delta^j_n\,\delta^k_o + \delta^i_m\,\delta^j_o\,\delta^k_n + \delta^i_n\,\delta^j_o\,\delta^k_m + \delta^i_n\,\delta^j_m\,\delta^k_o + \delta^i_o\,\delta^j_m\,\delta^k_n + \delta^i_o\,\delta^j_n\,\delta^k_m\right) \ , \\
[P_B]^{ijk}_{mno} &\equiv \frac{1}{6}\,\varepsilon^{jkt}\left(2\,\delta^i_m\,\varepsilon_{not} + \delta^i_n\,\varepsilon_{mot} + \delta^i_o\,\varepsilon_{nmt}\right) \ , \\
[P_C]^{ijk}_{mno} &\equiv \frac{1}{6}\left[\varepsilon^{ikt}\left(\delta^j_n\,\varepsilon_{mot} + \delta^j_o\,\varepsilon_{mnt}\right) + \varepsilon^{ijt}\left(\delta^k_n\,\varepsilon_{mot} + \delta^k_o\,\varepsilon_{mnt}\right)\right] \ , \\
[P_D]^{ijk}_{mno} &\equiv \frac{1}{6}\,\varepsilon^{ijk}\,\varepsilon_{mno} \,.
\end{aligned}$$

These are projectors for the reduction $\mathbf{3}\otimes\mathbf{3}\otimes\mathbf{3} = \mathbf{10}\oplus\mathbf{8}\oplus\mathbf{8}\oplus\mathbf{1}$.

(a) Verify that $[P_A]^{ijk}_{mno} + [P_B]^{ijk}_{mno} + [P_C]^{ijk}_{mno} + [P_D]^{ijk}_{mno} = \delta^i_m\,\delta^j_n\,\delta^k_o$.

(b) What are the dimensions of the projected representations?

(c) Carry out the same reduction with weight diagrams.

Exercise 4: Let T^a be the generators of isospin-SU(2) (cf. exercise 5.2) and $\hat{U}_I$ the time evolution operator, with the property $[T^a, \hat{U}_I] = 0$. Show that the expectation values

$$\langle I, I_3|\hat{U}_I|I, I_3\rangle$$

are independent of I_3.