Exercise 1: Following the lectures, an invariance group with the "metric" η (whereby $\eta^\dagger = \eta$, and η^{-1} exists) is composed of the elements $g \in G$ satisfying $g^\dagger \eta g = \eta$. The generators of G are T^a, and they satisfy $(T^a)^\dagger\,\eta = \eta\,T^a$, with $a = 1, \ldots, \text{dim}$. We now consider a vector space spanned by T^a, with elements of the form $v = \sum_{a=1}^{\text{dim}} v^a T^a$, with $v^a \in \mathbb{R}$. The adjoint representation operates in this vector space, and is defined through the mapping $g \mapsto D(g)$, with $v' \equiv [D(g)](v) \equiv g v g^{-1}$.

(a) Show that $v^\dagger = \eta v \eta^{-1}$ and that $D(g)$ respects this property.

(b) Consider a "small" transformation, $g = \exp\left(i \sum_{a=1}^{\text{dim}} \theta^a T^a\right)$, with $\theta^a \ll 1$. The generators F^a of the adjoint representation can be identified from the series expansion

$$(v')^b = v^b + \sum_{a,c=1}^{\text{dim}} i\,\theta^a (F^a)^{bc} v^c + \mathscr{O}(\theta^2)\ .$$

Verify that $(F^a)^{bc} = -i f^{abc}$.

(c) Making use of the Jacobi identity, show that $[F^a, F^b] = i f^{abc} F^c$.

Exercise 2: Let d be the dimension of a representation, T the normalization of the generators in this representation ($\text{tr}\{T^a T^b\} \equiv T\,\delta^{ab}$), and C a quadratic Casimir constant, defined as $\sum_{a=1}^{\text{dim}} (T^a T^a)_{AB} \equiv C\,\delta_{AB}$. Show that in SU($n$), the fundamental (F) and adjoint (A) representations have

$$d_\text{F} = n\,, \qquad T_\text{F} = \frac{1}{2}\,, \qquad C_\text{F} = \frac{n^2-1}{2n}\,,$$

$$d_\text{A} = n^2 - 1\,, \qquad T_\text{A} = n\,, \qquad C_\text{A} = n\,.$$

Exercise 3: Verify the equivalence of the irreducible representations $\mathbf{2}$ and $\mathbf{2}^*$ of SU(2). [*Hint:* use the Pauli matrix σ^2 as the similarity transformation.]

Exercise 4: Consider a matrix v of the form in Exercise 1. For the case of SU(2), verify the relations $\text{tr}[v^3] = 0$ and $\text{tr}[v^4] = (\text{tr}[v^2])^2/2$.